Computational Geometry

Convex Hulls in 3D
Lecture #9

[Comp. Geom A&A : Chapter 11]
**Complexity of the Convex Hull**

Given set $S$ of $n$ points in $\mathbb{R}^d$, what is max. $\#$ edges on $\partial \text{CH}(S)$?

<table>
<thead>
<tr>
<th>dim</th>
<th>w-c complexity of CH($S$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2 \in \Theta(1)$</td>
</tr>
<tr>
<td>2</td>
<td>$n \in \Theta(n)$</td>
</tr>
<tr>
<td>3</td>
<td>$3n - 6 \in \Theta(n)$</td>
</tr>
<tr>
<td>$d$</td>
<td>$\Theta(n^\lfloor d/2 \rfloor)$</td>
</tr>
</tbody>
</table>

**Upper Bound Theorem**

**Construction**

randomized-incremental!
Visibility

Face \( f \) is visible from \( p \) but not from \( q \).

Define conflict graph \( G \):

conflicts (visibility)

points

facets

\[ P_{\text{conflict}}(f) \]

\[ F_{\text{conflict}}(r) \]
Rand3dConvexHull(\(P \subset \mathbb{R}^3\))

pick non-coplanar set \(P' = \{p_1, \ldots, p_4\} \subseteq P\)

\(C \leftarrow \text{CH}(P')\)

compute rand. perm. \((p_5, \ldots, p_n)\) of \(P \setminus P'\)

initialize conflict graph \(G\)

\[
\begin{array}{l}
\text{for } r = 5 \text{ to } n \text{ do} \\
\quad \text{if } F_{\text{conflict}}(p_r) \neq \emptyset \text{ then } \{ p_r \notin C \} \\
\quad \text{delete all facets in } F_{\text{conflict}}(p_r) \text{ from } C \\
\quad \mathcal{L} \leftarrow \text{list of horizon edges visible from } p_r \\
\quad \text{foreach } e \in \mathcal{L} \text{ do} \\
\quad \\
\quad \quad f \leftarrow C.\text{create facet}(e, p_r); \text{ create vtx for } f \text{ in } G \\
\quad \quad (f_1, f_2) \leftarrow \text{previously_incident}_C(e) \\
\quad \quad P(e) \leftarrow P_{\text{conflict}}(f_1) \cup P_{\text{conflict}}(f_2) \\
\quad \quad \text{foreach } p \in P(e) \text{ do} \\
\quad \quad \quad \text{if } f \text{ is visible from } p \text{ then add edge } (p, f) \text{ to } G \\
\quad \quad \text{delete vtc } \{p_r\} \cup F_{\text{conflict}}(p_r) \text{ from } G \\
\end{array}
\]

return \(C\)

Worst-case running time \(= O(n^3)\)
Analysis

**Idea:** Bound expected *structural change*, that is, the total \#facets created by the algorithm.

**Lemma.** The expected \#facets created is at most $6n - 20$.

**Proof.**

\[
E[\#\text{facets created}] = 4 + \sum_{r=5}^{n} E[\#\text{facets incident to } p_r \text{ in } \text{CH}(P_r)] 
\leq 6n - 20
\]

For $r > 4$:

\[
E[\deg(p_r, \text{CH}(P_r))] = \frac{1}{r-4} \sum_{i=5}^{r} \deg(p_i, \text{CH}(P_r))
\leq \frac{1}{r-4} \left[ (\sum_{i=1}^{r} \deg(p_i)) - 12 \right]
\leq \frac{1}{r-4} \left[ 2 \cdot (3r - 6) - 12 \right] \leq 6
\]
**Running Time**

**Theorem:** The convex hull of a set of \( n \) pts in \( \mathbb{R}^3 \) can be computed in \( O(n \log n) \) expected time.

\[
\text{Rand3dConvexHull}(P \subset \mathbb{R}^3) \quad \begin{cases} 
\text{pick set } P' = \{p_1, \ldots, p_4\} \subseteq P \text{ of 4 non-coplanar pts} \\
\text{compute a random permutation } (p_5, \ldots, p_n) \text{ of } P \setminus P' \\
\text{initialize conflict graph } G \\
\text{for } r = 5 \text{ to } n \text{ do} \\
\quad \text{if } F_{\text{conflict}}(p_r) \neq \emptyset \text{ then} \\
\quad \quad \text{delete all facets in } F_{\text{conflict}}(p_r) \text{ from } C \\
\quad \quad \mathcal{L} \leftarrow \text{list of horizon edges visible from } p_r \\
\quad \text{foreach } e \in \mathcal{L} \text{ do} \\
\quad \quad \quad f \leftarrow C.\text{create facet}(e, p_r); \text{create vtx for } f \text{ in } G \\
\quad \quad \quad (f_1, f_2) \leftarrow \text{previously incident}_C(e) \\
\quad \quad \quad P(e) \leftarrow P_{\text{conflict}}(f_1) \cup P_{\text{conflict}}(f_2) \\
\quad \quad \text{foreach } p \in P(e) \text{ do} \\
\quad \quad \quad \text{if } f \text{ visible from } p \text{ then add edge } (p, f) \text{ to } G \\
\quad \text{delete vtx } \{p_r\} \cup F_{\text{conflict}}(p_r) \text{ from } G \\
\text{return } C
\end{cases}
\]

Stage \( r \) of for-loop (w/o outer foreach loop) takes time \( O(|F_{\text{conflict}}(p_r)|) = O(\#\text{facets deleted when adding } p_r) \)

This part of for-loop in total:
\[
E[\#\text{facets deleted}] = \leq E[\#\text{facets created}] = O(n).
\]

Outer foreach-loop:
- in stage \( r \): \( O(\sum_{e \in \mathcal{L}} |P(e)|) \)
- in total:
\[
O\left(\sum_{e \text{ on horizon at some moment}} |P(e)|\right) = O(n \log n)
\]

using configuration spaces, Section 9.5 [De Berg et al.]
Running times – expected vs. worst case

**Theorem:** The convex hull of a set of $n$ pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

**Exercise:** Give a simple deterministic algorithm that computes the convex hull in $O(n^2)$ (worst-case) time.
Define duality $\star$ between pts and (non-vertical) lines:

For $p = (p_x, p_y)$, define the line $p^\star: y = p_x x - p_y$.

For $\ell: y = mx + b$, define $\ell^\star$ to be the pt $q$ with $q^\star = \ell$, that is, $\ell^\star = (m, -b)$.

Observe:
- upper convex hulls of pts $\leftrightarrow$ lower envelopes of lines
- can compute intersections of “lower/upper” half planes (spaces) via upper/lower convex hulls
Voronoi Diagrams Revisited

Let $U : z = x^2 + y^2$ be the unit paraboloid in $\mathbb{R}^3$.

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Let $p = (p_x, p_y, 0)$.

Note that $p' \in h(p)$.

$h(p) : z = (2p_x)x + (2p_y)y - (p_x^2 + p_y^2)$

Note that $p' \in h(p)$.

$h(p) \cap U = \{p'\} \Rightarrow h(p)$ is tangent to $U$ (in $p'$)
Theorem: Let \( P \subset \mathbb{R}^2 \times \{0\} \) and \( \mathcal{H} = \{ h(p) \mid p \in P \} \).

Let \( \mathcal{E}(\mathcal{H}) \) be the upper envelope of \( \mathcal{H} \).

The projection of \( \mathcal{E}(\mathcal{H}) \) on \( z = 0 \) is \( \text{Vor}(P) \).

\[ \text{Vor}(P) \]

Exercise 11.10

Can compute \( \text{Vor}(P) \) in \( \mathbb{R}^2 \) via upper envelope in \( \mathbb{R}^3 \)

Upper envelope in \( \mathbb{R}^3 \) is in one-to-one correspondence to lower convex hull of the pt set \( \mathcal{H}^* \)

Use algorithm \text{Rand3dConvexHull}!