Computational Geometry

Height Interpolation
Lecture #8

[Comp. Geom A&A : Chapter 9]
Height Interpolation
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Triangulation of Planar Point Sets

**Definition:** Given $P \subseteq \mathbb{R}^2$, a *triangulation* of $P$ is a maximal planar subdivision with vtx set $P$, that is, no edge can be added without crossing.
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**Observe:** • all inner faces are triangles
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**Theorem:** Let $P \subset \mathbb{R}^2$ be a set of $n$ sites, not all collinear, and let $h$ be the number of sites on $\partial \text{CH}(P)$. 
Triangulation of Planar Point Sets

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**Theorem:** Let $P \subset \mathbb{R}^2$ be a set of $n$ sites, not all collinear, and let $h$ be the number of sites on $\partial \text{CH}(P)$. Then any triangulation of $P$ has $t(n, h)$ triangles and $e(n, h)$ edges.
**Triangulation of Planar Point Sets**

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**Theorem:** Let $P \subset \mathbb{R}^2$ be a set of $n$ sites, not all collinear, and let $h$ be the number of sites on $\partial \text{CH}(P)$. Then any triangulation of $P$ has $t(n, h)$ triangles and $e(n, h)$ edges. **Task:** Compute $t$ and $e$!
Back to Height Interpolation
Back to Height Interpolation
Back to Height Interpolation
Back to Height Interpolation
Back to Height Interpolation

height = 985

height = 23
Back to Height Interpolation

Intuition: Avoid “skinny” triangles!
Back to Height Interpolation

Intuition: Avoid “skinny” triangles!
In other words: avoid small angles!
Angle-Optimal Triangulations

**Definition:** Given a set $P \subset \mathbb{R}^2$
Angle-Optimal Triangulations

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![Triangulation Diagram]
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\[ \mathcal{T} \quad A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ) \]
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We say $A(\mathcal{T}) > A(\mathcal{T}')$.

![Diagram](image-url)
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We say \( A(\mathcal{T}) > A(\mathcal{T}') \)

\[ A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ) \]

\[ A(\mathcal{T}') = (30^\circ, 30^\circ, 30^\circ, 30^\circ, 120^\circ, 120^\circ) \]
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We say $A(\mathcal{T}) > A(\mathcal{T}')$ if $\exists i \in \{1, \ldots, 3m\}$:

$A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$

$A(\mathcal{T}') = (30^\circ, 30^\circ, 30^\circ, 30^\circ, 120^\circ, 120^\circ)$
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![Diagram](image.png)

$A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$

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We say \( A(\mathcal{T}) > A(\mathcal{T}') \) if \( \exists i \in \{1, \ldots, 3m\} : \alpha_i > \alpha_i' \) and

\[ A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ) \]

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We say $A(\mathcal{T}) > A(\mathcal{T}')$ if $\exists i \in \{1, \ldots, 3m\}$: $\alpha_i > \alpha'_i$ and $\forall j < i : \alpha_j = \alpha'_j$.

\[
A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ) \\
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**Angle-Optimal Triangulations**

**Definition:** Given a set $P \subset \mathbb{R}^2$ and a triangulation $\mathcal{T}$ of $P$, let $m$ be the number of triangles in $\mathcal{T}$ and let $A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$ be the **angle vector** of $\mathcal{T}$, where $\alpha_1 \leq \cdots \leq \alpha_{3m}$ are the angles in the triangles of $\mathcal{T}$.

We say $A(\mathcal{T}) > A(\mathcal{T}')$ if $\exists i \in \{1, \ldots, 3m\}: \alpha_i > \alpha'_i$ and $\forall j < i : \alpha_j = \alpha'_j$.

$\mathcal{T}$ is **angle-optimal** if $A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$, whereas $A(\mathcal{T}') = (30^\circ, 30^\circ, 30^\circ, 30^\circ, 120^\circ, 120^\circ)$.
Angle-Optimal Triangulations

**Definition:** Given a set $P \subset \mathbb{R}^2$ and a triangulation $\mathcal{T}$ of $P$, let $m$ be the number of triangles in $\mathcal{T}$ and let $A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$ be the *angle vector* of $\mathcal{T}$, where $\alpha_1 \leq \cdots \leq \alpha_{3m}$ are the angles in the triangles of $\mathcal{T}$.

We say $A(\mathcal{T}) > A(\mathcal{T}')$ if $\exists i \in \{1, \ldots, 3m\}$: $\alpha_i > \alpha_i'$ and $\forall j < i$: $\alpha_j = \alpha_j'$.

$\mathcal{T}$ is *angle-optimal* if $A(\mathcal{T}) \geq A(\mathcal{T}')$ for all triangulations $\mathcal{T}'$ of $P$.

$A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$

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Edge Flips

Definition: A triangulation $\mathcal{T}$ a triangulation. An edge $e$ of $\mathcal{T}$ is **illegal** if the minimum angle in the two triangles adjacent to $e$ increases when flipping.
Edge Flips

**Definition:** \( \mathcal{T} \) a triangulation. An edge \( e \) of \( \mathcal{T} \) is *illegal* if the minimum angle in the two triangles adjacent to \( e \) increases when flipping.

\[ \min_i \alpha_i = 30^\circ \]
Edge Flips

**Definition:** A triangulation $\mathcal{T}$ is a triangulation. An edge $e$ of $\mathcal{T}$ is illegal if the minimum angle in the two triangles adjacent to $e$ increases when flipping.

$\min_i \alpha_i = 30^\circ$

[Diagram showing a flip in a triangulation with $\min_i \alpha_i = 30^\circ$]
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**Definition:** A triangulation $\mathcal{T}$ is a triangulation. An edge $e$ of $\mathcal{T}$ is illegal if the minimum angle in the two triangles adjacent to $e$ increases when flipping.

**Observe:** Let $e$ be an illegal edge of $\mathcal{T}$, and $\mathcal{T}' = \text{flip}(\mathcal{T}, e)$.

\[
\min_i \alpha_i = 60^\circ \quad \text{and} \quad \min_i \alpha_i = 30^\circ
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**Observe:** Let $e$ be an illegal edge of $\mathcal{T}$, and $\mathcal{T}' = \text{flip}(\mathcal{T}, e)$. Then $A(\mathcal{T}') > A(\mathcal{T})$.

$$\min_i \alpha_i = \alpha_i = 60^\circ$$

$\mathcal{T}'$

flip

$\mathcal{T}$

$$\min_i \alpha_i = \alpha_i = 30^\circ$$
This is all Greek to me...

Theorem:
This is all Greek to me...

**Theorem:** (Thales)
The diameter of a circle always subtends a right angle to any point on the circle.
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\[ \{a, b\} := \ell \cap \partial D \ (a \neq b) \]
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\[ p, q \in \partial D \]
\[ \angle apb = \angle aqb \]
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$p, q \in \partial D$

$r \in \text{int}(D)$

$\angle apb = \angle aqb$
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\[ \angle apb = \angle aqb < \angle arb \]
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p, q \in \partial D
\]
\[
r \in \text{int}(D)
\]
\[
s \notin D
\]

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The diameter of a circle always subtends a right angle to any point on the circle.

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\[ r \in \text{int}(D) \]

\[ s \notin D \]

\[ \angle asb < \angle apb = \angle aqb < \angle arb \]
Legal Triangulations

**Lemma:** Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$. 

![Diagram showing a triangle $\Delta prq$ and a point $s$ inside the triangle, with $p$, $q$, and $r$ on the boundary $\partial D$.]
Lemma: Let $\Delta prq, \Delta pqs \in T$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$.

If $p, q, r, s$ in convex position and $s \notin \partial D$, then either $pq$ or $rs$ is illegal.
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**Lemma:** Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$.

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**Proof:**
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If $p, q, r, s$ in convex position and $s \notin \partial D$, then either $pq$ or $rs$ is illegal.

**Proof:** Show: \( \forall \alpha' \in \mathcal{T}' \ \exists \alpha \in \mathcal{T} \text{ s.t. } \alpha < \alpha' \).
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**Proof:** Show: $\forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T}$ s.t. $\alpha < \alpha'$. ("\(\Rightarrow\)"")
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Proof: Show: $\forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T}$ s.t. $\alpha < \alpha'$.

("$\Rightarrow$")
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**Proof:** Show: $\forall \alpha' \in T' \exists \alpha \in T$ s.t. $\alpha < \alpha'$.

(“$\Rightarrow$”)
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Lemma: Let $\triangle prq, \triangle pqs \in \mathcal{T}$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$.

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Proof: Show: $\forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T}$ s.t. $\alpha < \alpha'$.
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(“⇒”)
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**Proof:** Show: $\forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T}$ s.t. $\alpha < \alpha'$. ("$\Rightarrow$"")

Use Thales++ w.r.t. $qs'$. 
Legal Triangulations

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Proof: Show: $\forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T}$ s.t. $\alpha < \alpha'$.

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**Legal Triangulations**

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**Proof:**

Show: $\forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T}$ s.t. $\alpha < \alpha'$.

\(\Rightarrow\) Use Thales++ w.r.t. $qs'$.

\(\square\)
Legal Triangulations

**Lemma:** Let \( \Delta prq, \Delta pq s \in \mathcal{T} \) and \( p, q, r \in \partial D \). Then edge \( pq \) is illegal iff \( s \in \text{int}(D) \).

If \( p, q, r, s \) in convex position and \( s \notin \partial D \), then either \( pq \) or \( rs \) is illegal.

**Proof:**

Show: \( \forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T} \) s.t. \(\alpha < \alpha'\). ("\( \Rightarrow \" ")

Use Thales++ w.r.t. \( qs' \). □

**Note:** Criterion symmetric in \( r \) and \( s \)
Legal Triangulations

**Lemma:** Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$.

If $p, q, r, s$ in convex position and $s \not\in \partial D$, then either $pq$ or $rs$ is illegal.

**Proof:** Show: $\forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T}$ s.t. $\alpha < \alpha'$.

Use Thales++ w.r.t. $qs'$.

**Note:** Criterion symmetric in $r$ and $s$.

$\Rightarrow$ if $s \in \partial D$, both $pq$ and $rs$ legal.
Legal Triangulations

Lemma: Let \( \Delta prq, \Delta pqs \in \mathcal{T} \) and \( p, q, r \in \partial D \). Then edge \( pq \) is illegal iff \( s \in \text{int}(D) \).

If \( p, q, r, s \) in convex position and \( s \notin \partial D \), then either \( pq \) or \( rs \) is illegal.

Proof: Show: \( \forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T} \text{ s.t. } \alpha < \alpha' \). (\( \Rightarrow \)) Use Thales++ w.r.t. \( qs' \).

Note: Criterion symmetric in \( r \) and \( s \)
\( \Rightarrow \) if \( s \in \partial D \), both \( pq \) and \( rs \) legal.

Definition: A triangulation is legal if it has no illegal edge.
Legal Triangulations

**Lemma:** Let $\Delta prq, \Delta pqs \in T$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$.

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**Proof:** Show: $\forall \alpha' \in T' \exists \alpha \in T$ s.t. $\alpha < \alpha'$. 
("⇒") Use Thales++ w.r.t. $qs'$.

**Note:** Criterion symmetric in $r$ and $s$.

⇒ if $s \in \partial D$, both $pq$ and $rs$ legal.

**Definition:** A triangulation is *legal* if it has no illegal edge.

**Existence?**
Legal Triangulations

**Lemma:** Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$.

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**Definition:** A triangulation is *legal* if it has no illegal edge.

**Existence?** Algorithm: Let $\mathcal{T}$ be any triangulation of $P$. While $\mathcal{T}$ has an illegal edge $e$, flip $e$. Return $\mathcal{T}$.
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Legal Triangulations

**Lemma:** Let \( \Delta prq, \Delta pqs \in \mathcal{T} \) and \( p, q, r \in \partial D \). Then edge \( pq \) is illegal iff \( s \in \text{int}(D) \).

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**Proof:** Show: \( \forall \alpha' \in \mathcal{T}' \exists \alpha \in \mathcal{T} \) s.t. \( \alpha < \alpha' \). ("\( \Rightarrow \)"") Use Thales++ w.r.t. \( qs' \).

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**Definition:** A triangulation is **legal** if it has no illegal edge.

**Existence?** Algorithm: Let \( \mathcal{T} \) be any triangulation of \( P \).
While \( \mathcal{T} \) has an illegal edge \( e \), flip \( e \). Return \( \mathcal{T} \).

\[ A(\mathcal{T}) \text{ goes up!} \quad \& \quad \#(\text{triangulations of } P) < \infty \]
Legal Triangulations

**Lemma:** Let $\Delta prq, \Delta pqs \in T$ and $p, q, r \in \partial D$. Then edge $pq$ is illegal iff $s \in \text{int}(D)$.

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**Definition:** A triangulation is **legal** if it has no illegal edge.

**Existence?** Algorithm: Let $T$ be any triangulation of $P$. While $T$ has an illegal edge $e$, flip $e$. Return $T$.

$A(T)$ goes up! $\&$ $\#$(triangulations of $P) < \infty$
Legal vs. Angle-Optimal

Clearly... Every angle-optimal triangulation is legal.
Legal vs. Angle-Optimal

Clearly... Every angle-optimal triangulation is legal.

But is every legal triangulation angle-optimal??
Legal vs. Angle-Optimal

Clearly... Every angle-optimal triangulation is legal.

*But is every legal triangulation angle-optimal??*

Let’s see.
Legal vs. Angle-Optimal

Clearly... Every angle-optimal triangulation is legal.

But is every legal triangulation angle-optimal??

Let’s see.

To clarify things, we’ll introduce yet another type of triangulation...
Voronoi & Delaunay

**Recall:**

Given a set $P$ of $n$ points in the plane...

$\text{Vor}(P) = \text{subdivision of the plane into Voronoi cells, edges, and vertices}$

$\mathcal{V}(p) = \{x \in \mathbb{R}^2 : |xp| < |xq| \text{ for all } q \in P \setminus \{p\}\}$

Voronoi cell of $p \in P$
Voronoi & Delaunay

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Voronoi cell of $p \in P$

**Definition:** The graph $G = (P, E)$ with

$\{p, q\} \in E \iff \mathcal{V}(p)$ and $\mathcal{V}(q)$ share an edge

is the *dual graph* of $\text{Vor}(P)$
Recall: Given a set \( P \) of \( n \) points in the plane...
\[ \text{Vor}(P) = \text{subdivision of the plane into Voronoi cells, edges, and vertices} \]
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Voronoi cell of \( p \in P \)

Definition: The graph \( \mathcal{G} = (P, E) \) with
\[ \{p, q\} \in E \iff \mathcal{V}(p) \text{ and } \mathcal{V}(q) \text{ share an edge} \]
is the dual graph of \( \text{Vor}(P) \)

Definition: The Delaunay graph \( \mathcal{DG}(P) \) is the straight-line drawing of \( \mathcal{G} \).
From Voronoi to Delaunay

\[ P \subset \mathbb{R}^2 \]
From Voronoi to Delaunay

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Georgy Feodosevich Voronoy
(1868–1908 Zhuravki, now Ukraine)
From Voronoi to Delaunay

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\[ P \subset \mathbb{R}^2 \]

\[ \text{Vor}(P) \]

\[ \mathcal{D}G(P) \]
From Voronoi to Delaunay

\[ P \subset \mathbb{R}^2 \]

Georgy Feodosevich Voronoy (1868–1908 Zhuravki, now Ukraine)

Boris Nikolaevich Delone (St. Petersburg 1890–1980 Moscow)

DG(P)

Vor(P)
Planarity

**Theorem.** \( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \) \( \mathcal{DG}(P) \) plane.
Planarity

**Theorem.** $P \subset \mathbb{R}^2$ finite $\Rightarrow DG(P)$ plane.

**Proof.** Recall property of Voronoi edges:
Planarity

**Theorem.** \( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \) \( \mathcal{DG}(P) \) plane.

**Proof.** Recall property of Voronoi edges:
Edge \( pq \) is in \( \mathcal{DG}(P) \) \( \iff \)
Planarity

**Theorem.** \( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \) \( \mathcal{DG}(P) \) plane.

**Proof.** Recall property of Voronoi edges:

Edge \( pq \) is in \( \mathcal{DG}(P) \) \( \Leftrightarrow \) \( \exists D_{pq} \) closed disk s.t.
Planarity

**Theorem.** \( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \) \( DG(P) \) plane.

**Proof.** Recall property of Voronoi edges:

Edge \( pq \) is in \( DG(P) \) \( \iff \exists D_{pq} \) closed disk s.t.
Planarity

**Theorem.** \( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \) \( DG(P) \) plane.

**Proof.** Recall property of Voronoi edges:
Edge \( pq \) is in \( DG(P) \) \( \iff \) \( \exists D_{pq} \) closed disk s.t.
- \( p, q \in \partial D_{pq} \) and
**Planarity**

**Theorem.** \( P \subset \mathbb{R}^2 \text{ finite } \Rightarrow \mathcal{DG}(P) \text{ plane.} \)

**Proof.**
Recall property of Voronoi edges:
Edge \( pq \) is in \( \mathcal{DG}(P) \) \( \iff \exists D_{pq} \text{ closed disk s.t.} \)
- \( p, q \in \partial D_{pq} \) and
- \( \{p, q\} = D_{pq} \cap P. \)
Planarity

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Edge \( pq \) is in \( \mathcal{DG}(P) \) \( \iff \exists D_{pq} \) closed disk s.t.

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\( c = \text{center}(D_{pq}) \) lies on edge betw. \( V(p) \) & \( V(q) \).
**Planarity**

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Suppose \( \exists \) edge \( uv \neq pq \) in \( \mathcal{DG}(P) \) that crosses \( pq \).
Planarity

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Edge \( pq \) is in \( DG(P) \) \( \iff \exists \) \( D_{pq} \) closed disk s.t.
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Suppose \( \exists \) edge \( uv \neq pq \) in \( DG(P) \) that crosses \( pq. \)

\( u, v \not\in D_{pq} \Rightarrow \)
Planarity

**Theorem.** \( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \) \( \mathcal{DG}(P) \) plane.

**Proof.** Recall property of Voronoi edges:

- Edge \( pq \) is in \( \mathcal{DG}(P) \) \( \iff \) \( \exists D_{pq} \) closed disk s.t.
  - \( p, q \in \partial D_{pq} \) and
  - \( \{p, q\} = D_{pq} \cap P \).

- \( c = \text{center}(D_{pq}) \) lies on edge betw. \( V(p) \) & \( V(q) \).

Suppose \( \exists \) edge \( uv \neq pq \) in \( \mathcal{DG}(P) \) that crosses \( pq \).

- \( u, v \notin D_{pq} \Rightarrow u, v \notin t_{pq} \Rightarrow \)
Planarity

**Theorem.** \( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \mathcal{DG}(P) \) plane.

**Proof.**

Recall property of Voronoi edges:

Edge \( pq \) is in \( \mathcal{DG}(P) \) \( \iff \exists D_{pq} \) closed disk s.t.

- \( p, q \in \partial D_{pq} \) and
- \( \{p, q\} = D_{pq} \cap P \).

\( c = \text{center}(D_{pq}) \) lies on edge betw. \( \mathcal{V}(p) \) & \( \mathcal{V}(q) \).

Suppose \( \exists \) edge \( uv \neq pq \) in \( \mathcal{DG}(P) \) that crosses \( pq \).

\( u, v \notin D_{pq} \Rightarrow u, v \notin t_{pq} \Rightarrow \)

\( uv \) crosses another edge of \( t_{pq} \)
Planarity

**Theorem.** \( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \mathcal{DG}(P) \) plane.

**Proof.**

Recall property of Voronoi edges:

Edge \( pq \) is in \( \mathcal{DG}(P) \) if and only if there exists a closed disk \( D_{pq} \) such that:

- \( p, q \in \partial D_{pq} \)
- \( \{p, q\} = D_{pq} \cap P \)
- \( c = \text{center}(D_{pq}) \) lies on the edge between \( \mathcal{V}(p) \) and \( \mathcal{V}(q) \).

Suppose there exists an edge \( uv \neq pq \) in \( \mathcal{DG}(P) \) that crosses \( pq \).

\( u, v \not\in D_{pq} \Rightarrow u, v \not\in t_{pq} \Rightarrow uv \) crosses another edge of \( t_{pq} \)

\( p, q \not\in D_{uv} \Rightarrow \)
Planarity

**Theorem.** $P \subset \mathbb{R}^2$ finite $\Rightarrow \mathcal{DG}(P)$ plane.

**Proof.**

Recall property of Voronoi edges:

Edge $pq$ is in $\mathcal{DG}(P) \iff \exists D_{pq}$ closed disk s.t.

- $p, q \in \partial D_{pq}$ and
- $\{p, q\} = D_{pq} \cap P$.

$c = \text{center}(D_{pq})$ lies on edge betw. $\mathcal{V}(p)$ & $\mathcal{V}(q)$.

Suppose $\exists$ edge $uv \neq pq$ in $\mathcal{DG}(P)$ that crosses $pq$.

$u, v \notin D_{pq} \Rightarrow u, v \notin t_{pq} \Rightarrow$

$uv$ crosses another edge of $t_{pq}$

$p, q \notin D_{uv} \Rightarrow p, q \notin t_{uv} \Rightarrow$
Planarity

**Theorem.** $P \subset \mathbb{R}^2$ finite $\Rightarrow$ $\mathcal{DG}(P)$ plane.

**Proof.** Recall property of Voronoi edges:

Edge $pq$ is in $\mathcal{DG}(P) \iff \exists D_{pq}$ closed disk s.t.

- $p, q \in \partial D_{pq}$ and
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Suppose $\exists$ edge $uv \neq pq$ in $\mathcal{DG}(P)$ that crosses $pq$.

$u, v \notin D_{pq} \Rightarrow u, v \notin t_{pq} \Rightarrow$

$uv$ crosses another edge of $t_{pq}$

$p, q \notin D_{uv} \Rightarrow p, q \notin t_{uv} \Rightarrow$

$pq$ crosses another edge of $t_{uv}$
Planarity

**Theorem.**  
\( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \) \( D\mathcal{G}(P) \) plane.

**Proof.**
Recall property of Voronoi edges:
Edge \( pq \) is in \( D\mathcal{G}(P) \) \( \iff \exists D_{pq} \) closed disk s.t.
- \( p, q \in \partial D_{pq} \) and
- \( \{p, q\} = D_{pq} \cap P. \)

\( c = \text{center}(D_{pq}) \) lies on edge betw. \( V(p) \) \& \( V(q) \).

Suppose \( \exists \) edge \( uv \neq pq \) in \( D\mathcal{G}(P) \) that crosses \( pq \).

\( u, v \notin D_{pq} \Rightarrow u, v \notin t_{pq} \Rightarrow \)
\( uv \) crosses another edge of \( t_{pq} \)
\( p, q \notin D_{uv} \Rightarrow p, q \notin t_{uv} \Rightarrow \)
\( pq \) crosses another edge of \( t_{uv} \)
\( \Rightarrow \) one of \( s_{pq} \) or \( s_{qp} \) crosses one of \( s_{uv} \) or \( s_{vu} \)
Planarity

**Theorem.** \( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \mathcal{D} \mathcal{G}(P) \) plane.

**Proof.**

Recall property of Voronoi edges:

Edge \( pq \) is in \( \mathcal{D} \mathcal{G}(P) \) \( \iff \exists \mathcal{D}_{pq} \) closed disk s.t.

- \( p, q \in \partial \mathcal{D}_{pq} \) and
- \( \{p, q\} = \mathcal{D}_{pq} \cap P \).

\( c = \text{center}(\mathcal{D}_{pq}) \) lies on edge betw. \( \mathcal{V}(p) \) & \( \mathcal{V}(q) \).

Suppose \( \exists \) edge \( uv \neq pq \) in \( \mathcal{D} \mathcal{G}(P) \) that crosses \( pq \).

\[ u, v \not\in \mathcal{D}_{pq} \Rightarrow u, v \not\in t_{pq} \Rightarrow \]

\( uv \) crosses another edge of \( t_{pq} \)

\[ p, q \not\in \mathcal{D}_{uv} \Rightarrow p, q \not\in t_{uv} \Rightarrow \]

\( pq \) crosses another edge of \( t_{uv} \)

\( \Rightarrow \) one of \( s_{pq} \) or \( s_{qp} \) crosses one of \( s_{uv} \) or \( s_{vu} \).
Planarity

**Theorem.** \( P \subset \mathbb{R}^2 \) finite \( \Rightarrow \mathcal{DG}(P) \) plane.

**Proof.**

Recall property of Voronoi edges:

Edge \( pq \) is in \( \mathcal{DG}(P) \) \( \iff \exists D_{pq} \) closed disk s.t.

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\( u, v \notin D_{pq} \Rightarrow u, v \notin t_{pq} \Rightarrow \)

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\( p, q \notin D_{uv} \Rightarrow p, q \notin t_{uv} \Rightarrow \)

\( pq \) crosses another edge of \( t_{uv} \)

\( \Rightarrow \) one of \( s_{pq} \) or \( s_{qp} \) crosses one of \( s_{uv} \) or \( s_{vu}. \)

\( s_{pq} \subset \mathcal{V}(p), s_{qp} \subset \mathcal{V}(q), s_{uv} \subset \mathcal{V}(u), s_{vu} \subset \mathcal{V}(v). \)
Planarity

**Theorem.** $P \subset \mathbb{R}^2$ finite $\Rightarrow \mathcal{DG}(P)$ plane.

**Proof.**

Recall property of Voronoi edges:

Edge $pq$ is in $\mathcal{DG}(P)$ $\iff$ $\exists D_{pq}$ closed disk s.t.

- $p, q \in \partial D_{pq}$ and
- $\{p, q\} = D_{pq} \cap P$.

$c = \text{center}(D_{pq})$ lies on edge betw. $\mathcal{V}(p) \& \mathcal{V}(q)$.

Suppose $\exists$ edge $uv \neq pq$ in $\mathcal{DG}(P)$ that crosses $pq$.

\[ u, v \not\in D_{pq} \Rightarrow u, v \not\in t_{pq} \Rightarrow \]

$uv$ crosses another edge of $t_{pq}$

\[ p, q \not\in D_{uv} \Rightarrow p, q \not\in t_{uv} \Rightarrow \]

$pq$ crosses another edge of $t_{uv}$

$\Rightarrow$ one of $s_{pq}$ or $s_{qp}$ crosses one of $s_{uv}$ or $s_{vu}$

$s_{pq} \subset \mathcal{V}(p)$, $s_{qp} \subset \mathcal{V}(q)$, $s_{uv} \subset \mathcal{V}(u)$, $s_{vu} \subset \mathcal{V}(v)$. 
Characterization

Characterization of Voronoi vertices and Voronoi edges ⇒

Theorem. \( P \subset \mathbb{R}^2 \) finite. Then

(i) Three pts \( p, q, r \in P \) are vertices of the same face of \( DG(P) \) \( \iff \) \( \text{int}(C(p, q, r)) \cap P = \emptyset \)
Characterization

Characterization of Voronoi vertices and Voronoi edges ⇒

**Theorem.** \( P \subset \mathbb{R}^2 \) finite. Then

(i) Three pts \( p, q, r \in P \) are vertices of the same face of \( \mathcal{DG}(P) \) ⇔ \( \text{int}(C(p, q, r)) \cap P = \emptyset \)

(ii) Two pts \( p, q \in P \) form an edge of \( \mathcal{DG}(P) \) ⇔ there is a disk \( D \) with

• \( \partial D \cap P = \{p, q\} \) and

• \( \text{int}(D) \cap P = \emptyset \).
Characterization

Characterization of Voronoi vertices and Voronoi edges \( \Rightarrow \)

**Theorem.** \( P \subset \mathbb{R}^2 \) finite. Then

(i) Three pts \( p, q, r \in P \) are vertices of the same face of \( \mathcal{DG}(P) \) \( \iff \) \( \text{int}(C(p, q, r)) \cap P = \emptyset \)

(ii) Two pts \( p, q \in P \) form an edge of \( \mathcal{DG}(P) \) \( \iff \)

\[
\text{there is a disk } D \text{ with } \begin{align*}
\bullet \ & \partial D \cap P = \{p, q\} \text{ and} \\
\bullet \ & \text{int}(D) \cap P = \emptyset.
\end{align*}
\]

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then

\( \mathcal{T} \) Delaunay \( \iff \) for each triangle \( \Delta \) of \( \mathcal{T} \):

\[
\text{int}(C(\Delta)) \cap P = \emptyset.
\]
Characterization

Characterization of Voronoi vertices and Voronoi edges ⇒

**Theorem.** \( P \subset \mathbb{R}^2 \) finite. Then

(i) Three pts \( p, q, r \in P \) are vertices of the same face of \( DG(P) \) ⇔ \( \text{int}(C(p, q, r)) \cap P = \emptyset \)

(ii) Two pts \( p, q \in P \) form an edge of \( DG(P) \) ⇔ there is a disk \( D \) with

\[ \bullet \ \partial D \cap P = \{ p, q \} \text{ and} \]
\[ \bullet \ \text{int}(D) \cap P = \emptyset. \]

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then

\( \mathcal{T} \) Delaunay ⇔ for each triangle \( \Delta \) of \( \mathcal{T} \):

\[ \text{int}(C(\Delta)) \cap P = \emptyset. \]

(“empty-circumcircle property”)
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \).

Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \) \( \mathcal{T} \) Delaunay.
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( T \) triangulation of \( P \).
Then \( T \) legal \( \iff \) \( T \) Delaunay.

**Proof.** “\( \Leftarrow \)"
Main Result

**Theorem.** $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of $P$. Then $\mathcal{T}$ legal $\iff \mathcal{T}$ Delaunay.

**Proof.** “$\Leftarrow$” implied by empty-circumcircle property & Thales++
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \).
Then \( \mathcal{T} \) legal \( \iff \) \( \mathcal{T} \) Delaunay.

**Proof.** “\( \Leftarrow \)” implied by empty-circumcircle property & Thales++

“\( \Rightarrow \)”
Main Result

**Theorem.** $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of $P$. Then $\mathcal{T}$ legal $\iff \mathcal{T}$ Delaunay.

**Proof.** “$\Leftarrow$” implied by empty-circumcircle property & Thales++

“$\Rightarrow$” by contradiction:
Main Result

Theorem. \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \).
Then \( \mathcal{T} \) legal \( \iff \) \( \mathcal{T} \) Delaunay.

Proof. “\( \iff \)” implied by empty-circumcircle property & Thales++
“\( \Rightarrow \)” by contradiction:
Assume \( \mathcal{T} \) is legal triang. of \( P \), but not Delaunay.
Main Result

**Theorem.** $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of $P$. Then $\mathcal{T}$ legal $\iff \mathcal{T}$ Delaunay.

**Proof.** “$\Leftarrow$” implied by empty-circumcircle property & Thales++

“$\Rightarrow$” by contradiction:

Assume $\mathcal{T}$ is legal triang. of $P$, but not Delaunay.

$\Rightarrow \exists \Delta pqr$ such that $\text{int}(C(\Delta pqr))$ contains $s \in P.$
Main Result

**Theorem.** $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of $P$.

Then $\mathcal{T}$ legal $\iff \mathcal{T}$ Delaunay.

**Proof.** “$\Leftarrow$” implied by empty-circumcircle property & Thales++

“$\Rightarrow$” by contradiction:

Assume $\mathcal{T}$ is legal triang. of $P$, but not Delaunay.

$\Rightarrow \exists \Delta pqr$ such that int($C(\Delta pqr)$) contains $s \in P$.

Wlog. let $e = pq$ be the edge of $\Delta pqr$ such that $s$ “sees” $pq$ before the other edges of $\Delta pqr$. 

![Diagram showing a triangle with labeled points and the circumcircle](attachment:diagram.png)
Main Result

Theorem. \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \).
Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

Proof. \( \iff \) implied by empty-circumcircle property \& Thales++

\( \implies \) by contradiction:
Assume \( \mathcal{T} \) is legal triang. of \( P \), but not Delaunay.
\( \implies \exists \Delta pqr \) such that \( \text{int}(C(\Delta pqr)) \) contains \( s \in P \).

Wlog. let \( e = pq \) be the edge of \( \Delta pqr \)
such that \( s \) “sees” \( pq \) before the other edges of \( \Delta pqr \).

Among all such pairs \((\Delta pqr, s)\) in \( \mathcal{T} \)
choose one that maximizes \( \alpha = \angle psq \).
Proof of Main Result (cont’d)

Consider the triangle $\triangle pqt$ adjacent to $e$ in $\mathcal{T}$. 
Proof of Main Result (cont’d)

Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$. $\mathcal{T}$ legal $\Rightarrow$
Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$.

$\mathcal{T}$ legal $\Rightarrow$ $e$ legal $\Rightarrow$
Proof of Main Result (cont’d)

Consider the triangle \( \Delta pqt \) adjacent to \( e \) in \( \mathcal{T} \).
\( \mathcal{T} \) legal \( \Rightarrow \) e legal \( \Rightarrow \) \( t \notin \text{int}(C(\Delta pqr)) \)
Proof of Main Result (cont’d)

Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$. $\mathcal{T}$ legal $\Rightarrow$ $e$ legal $\Rightarrow$ $t \not\in \text{int}(C(\Delta pqr))$ $\Rightarrow$ $C(\Delta pqt)$ contains $C(\Delta pqr) \cap e^+$. 
Proof of Main Result (cont’d)

Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$. 
$\mathcal{T}$ legal $\Rightarrow$ $e$ legal $\Rightarrow$ $t \notin \text{int}(C(\Delta pqr))$
$\Rightarrow C(\Delta pqt)$ contains $C(\Delta pqr) \cap e^+$. 

\begin{align*}
\text{halfplane} & \quad \text{supported by } e \\
\text{that contains } s
\end{align*}
Proof of Main Result (cont’d)

Consider the triangle \( \Delta pqt \) adjacent to \( e \) in \( \mathcal{T} \).

\( \mathcal{T} \) legal \( \Rightarrow \) e legal \( \Rightarrow \) \( t \notin \text{int}(C(\Delta pqr)) \)

\( \Rightarrow \) \( C(\Delta pqt) \) contains \( C(\Delta pqr) \cap e^+ \).

\( \Rightarrow \) \( s \in C(\Delta pqt) \)

\( \Rightarrow \) s ∈ \( C(\Delta pqt) \)

The halfplane supported by e that contains s
Proof of Main Result (cont’d)

Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$.

$\mathcal{T}$ legal $\Rightarrow$ $e$ legal $\Rightarrow$ $t \notin \text{int}(C(\Delta pqr))$

$\Rightarrow C(\Delta pqt)$ contains $C(\Delta pqr) \cap e^+$.  

$\Rightarrow s \in C(\Delta pqt)$

Wlog. let $e' = qt$ be the edge of $\Delta pqt$ that $s$ sees.
Proof of Main Result (cont’d)

Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$.

$\mathcal{T}$ legal $\Rightarrow$ $e$ legal $\Rightarrow$ $t \notin \text{int}(C(\Delta pqr))$

$\Rightarrow C(\Delta pqt)$ contains $C(\Delta pqr) \cap e^\perp$.

$\Rightarrow s \in C(\Delta pqt)$

Wlog. let $e' = qt$ be the edge of $\Delta pqt$ that $s$ sees.

$\Rightarrow \beta = \angle tsq > \alpha = \angle psq$
Proof of Main Result (cont’d)

Consider the triangle \( \triangle pqt \) adjacent to \( e \) in \( \mathcal{T} \).

\( \mathcal{T} \) legal \( \Rightarrow \) \( e \) legal \( \Rightarrow \) \( t \not\in \text{int}(C(\Delta pqr)) \)

\( \Rightarrow \) \( C(\Delta pqt) \) contains \( C(\Delta pqr) \cap e^+ \).

\( \Rightarrow s \in C(\Delta pqt) \)

Wlog. let \( e' = qt \) be the edge of \( \Delta pqt \) that \( s \) sees.

\( \Rightarrow \beta = \angle tsq > \alpha = \angle psq \)
Proof of Main Result (cont’d)

Consider the triangle $\Delta pqt$ adjacent to $e$ in $\mathcal{T}$. $\mathcal{T}$ legal $\Rightarrow$ $e$ legal $\Rightarrow$ $t \notin \text{int}(C(\Delta pqr))$ $\Rightarrow$ $C(\Delta pqt)$ contains $C(\Delta pqr) \cap e^+$. $\Rightarrow$ $s \in C(\Delta pqt)$

Wlog. let $e' = qt$ be the edge of $\Delta pqt$ that $s$ sees. $\Rightarrow \beta = \angle tsq > \alpha = \angle psq$

Contradiction to choice of the pair $\langle \Delta pqr, s \rangle$. □
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \).
Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.
Main Result

**Theorem.** $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of $P$. Then $\mathcal{T}$ legal $\iff \mathcal{T}$ Delaunay.

**Observation.** Suppose $P$ is in general position...
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position. . . no 4 pts on an empty circle!
Main Result

**Theorem.** $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of $P$. Then $\mathcal{T}$ legal $\iff$ $\mathcal{T}$ Delaunay.

**Observation.** Suppose $P$ is in general position. ... 
$\Rightarrow$ Delaunay triangulation unique
Main Result

Theorem.  \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

Observation.  Suppose \( P \) is in general position. . .

\[ \Rightarrow \text{Delaunay triangulation unique } \left[ \mathcal{DG}(P)! \right] \]
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position. . .
\[\Rightarrow\] Delaunay triangulation unique \[DG(P)\] no 4 pts on an empty circle!
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \).
Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position... ⇒ Delaunay triangulation unique  
⇒ legal triangulation unique  
\[ \Downarrow \]  
\noindent \textcolor{yellow}{\textbf{no 4 pts on an empty circle!}}
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position . . .

\[ \Rightarrow \text{Delaunay triangulation unique} \quad [\mathcal{D}(P)!] \]

\[ \Rightarrow \text{legal triangulation unique} \]

\[ \Downarrow \quad \text{angle-optimal} \Rightarrow \text{legal} \]
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \).
Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position.
\( \Rightarrow \) Delaunay triangulation unique
\( \Rightarrow \) legal triangulation unique
\( \Downarrow \) angle-optimal \( \Rightarrow \) legal  
[by def.]

no 4 pts on an empty circle!

\( \mathcal{DG}(P)! \)
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \). Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position.

\[ \Rightarrow \] Delaunay triangulation unique \[ [\mathcal{DG}(P)!] \]

\[ \Rightarrow \] legal triangulation unique

\[ \Downarrow \] angle-optimal \( \Rightarrow \) legal \[ [\text{by def.}] \]

Delaunay triangulation is angle-optimal!
Main Result

**Theorem.** $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of $P$. Then $\mathcal{T}$ legal $\iff \mathcal{T}$ Delaunay.

**Observation.** Suppose $P$ is in general position. . .

⇒ Delaunay triangulation unique $[\mathcal{DG}(P)!]$ 
⇒ legal triangulation unique 

$\Downarrow$ angle-optimal $\Rightarrow$ legal $[\text{by def.}]$

Delaunay triangulation is angle-optimal!

Suppose $P$ is not in general position. . .

no 4 pts on an empty circle!
Main Result

**Theorem.** $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of $P$. Then $\mathcal{T}$ legal $\iff$ $\mathcal{T}$ Delaunay.

**Observation.** Suppose $P$ is in general position. . .

$\Rightarrow$ Delaunay triangulation unique $[\mathcal{DG}(P)!]$

$\Rightarrow$ legal triangulation unique

$\Downarrow$ angle-optimal $\Rightarrow$ legal [by def.]

Delaunay triangulation is angle-optimal!

Suppose $P$ is not in general position. . .

$\Rightarrow$ Delaunay graph has convex “holes” bounded by co-circular pts

no 4 pts on an empty circle!
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \).
Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position.
\[ \Rightarrow \text{Delaunay triangulation unique} \quad [\mathcal{DG}(P)!] \]
\[ \Rightarrow \text{legal triangulation unique} \]
\[ \Downarrow \text{angle-optimal} \Rightarrow \text{legal} \quad [\text{by def.}] \]
Delaunay triangulation is angle-optimal!

Suppose \( P \) is not in general position.
\[ \Rightarrow \text{Delaunay graph has convex “holes”} \]
bounded by co-circular pts
\[ \Downarrow \text{Thales++} \quad \text{homework exercise!} \]
Main Result

**Theorem.** \( P \subset \mathbb{R}^2 \) finite, \( \mathcal{T} \) triangulation of \( P \).
Then \( \mathcal{T} \) legal \( \iff \mathcal{T} \) Delaunay.

**Observation.** Suppose \( P \) is in general position.
\( \Rightarrow \) Delaunay triangulation unique [\( \mathcal{DG}(P)! \)]
\( \Rightarrow \) legal triangulation unique
\( \Downarrow \) angle-optimal \( \Rightarrow \) legal [by def.]
Delaunay triangulation is angle-optimal!

Suppose \( P \) is not in general position.
\( \Rightarrow \) Delaunay graph has convex “holes” bounded by co-circular pts
\( \Downarrow \) Thales++
All Delaunay triang. have same min. angle.
Computation

Fact. A Delaunay triangulation of an arbitrary set of $n$ pts in the plane can be computed in $O(n \log n)$ time.
Computation

Fact. A Delaunay triangulation of an arbitrary set of $n$ pts in the plane can be computed in $O(n \log n)$ time. [Compute dual of Vor($P$), fill holes.]
Computation

**Fact.**  A Delaunay triangulation of an arbitrary set of $n$ pts in the plane can be computed in $O(n \log n)$ time.  

[Compute dual of Vor($P$), fill holes.]

**Corollary.**  An angle-optimal triangulation of a set of $n$ pts in general position can be computed in $O(n \log n)$ time.
Computation

**Fact.** A Delaunay triangulation of an arbitrary set of $n$ pts in the plane can be computed in $O(n \log n)$ time. [Compute dual of Vor($P$), fill holes.]

**Corollary.** An angle-optimal triangulation of a set of $n$ pts in general position can be computed in $O(n \log n)$ time. [DG!]
Computation

Fact.  A Delaunay triangulation of an arbitrary set of \( n \) pts in the plane can be computed in \( O(n \log n) \) time.  
[Compute dual of Vor(\( P \)), fill holes.]

Corollary.  An angle-optimal triangulation of a set of \( n \) pts in general position can be computed in \( O(n \log n) \) time.

Given an arbitrary set of \( n \) pts, a triangulation maximizing the minimum angle can be computed in \( O(n \log n) \) time.
**Computation**

**Fact.** A Delaunay triangulation of an arbitrary set of $n$ pts in the plane can be computed in $O(n \log n)$ time. \[ \text{[Compute dual of Vor}(P)\text{, fill holes.]} \]

**Corollary.** An angle-optimal triangulation of a set of $n$ pts in general position can be computed in $O(n \log n)$ time.

Given an arbitrary set of $n$ pts, a triangulation maximizing the minimum angle can be computed in $O(n \log n)$ time. \[ \text{[DG!]} \]

[Use fact.]
Computation

**Fact.** A Delaunay triangulation of an arbitrary set of \( n \) pts in the plane can be computed in \( O(n \log n) \) time. [Compute dual of \( \text{Vor}(P) \), fill holes.]

**Corollary.** An angle-optimal triangulation of a set of \( n \) pts in general position can be computed in \( O(n \log n) \) time.

Given an arbitrary set of \( n \) pts, a triangulation maximizing the minimum angle can be computed in \( O(n \log n) \) time. [Use fact.]

An angle-optimal triangulation of an arbitrary set of \( n \) pts can be computed in \( O(n^2) \) time.
Computation

**Fact.** A Delaunay triangulation of an arbitrary set of $n$ pts in the plane can be computed in $O(n \log n)$ time. [Compute dual of Vor($P$), fill holes.]

**Corollary.** An angle-optimal triangulation of a set of $n$ pts in general position can be computed in $O(n \log n)$ time.

Given an arbitrary set of $n$ pts, a triangulation maximizing the minimum angle can be computed in $O(n \log n)$ time. [Use fact.]

An angle-optimal triangulation of an arbitrary set of $n$ pts can be computed in $O(n^2)$ time. [How?]