Computational Geometry

Point Localization

or

Where am I?

Lecture #6

[Comp. Geom A&A : Chapter 6]
What’s the Problem?
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**Task:** Given a planar subdivision $S$ with $n$ segments, preprocess $S$ to allow for fast point location queries!
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**Solution:** Pre-proc: Partition $S$ into slabs induced by vertices.
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Query:
What’s the Problem?

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Query: – find right slab
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Query: – find right slab
– search slab
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**Task:** Given a planar subdivision $\mathcal{S}$ with $n$ segments, preprocess $\mathcal{S}$ to allow for fast point location queries!

**Solution:** Pre-proc: Partition $\mathcal{S}$ into slabs induced by vertices.

Query: $\begin{cases} \text{find right slab} \\ \text{search slab} \end{cases}$ $\implies$ 2 bin. searches!
What’s the Problem?

Task: Given a planar subdivision $S$ with $n$ segments, preprocess $S$ to allow for fast point location queries!

Solution: Pre-proc: Partition $S$ into slabs induced by vertices.

Query: 
- find right slab
- search slab \[ \{ \text{2 bin. searches!} \] \[ O(\log n) \] time!}
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Task: Given a planar subdivision $S$ with $n$ segments, preprocess $S$ to allow for fast point location queries!

Solution: Pre-proc: Partition $S$ into slabs induced by vertices.

Query: \[ \begin{align*}
&\text{– find right slab} \\
&\text{– search slab}
\end{align*} \] \{ 2 bin. searches! \} \[ O(\log n) \] time!

But:
What’s the Problem?

Task: Given a planar subdivision $S$ with $n$ segments, preprocess $S$ to allow for fast point location queries!

Solution: Pre-proc: Partition $S$ into slabs induced by vertices.

Query: 
- find right slab
- search slab

$\{2 \text{ bin. searches!} \}$

$O(\log n)$ time!

But: Space?
What’s the Problem?

**Task:** Given a planar subdivision $S$ with $n$ segments, preprocess $S$ to allow for fast point location queries!

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Query: – find right slab
– search slab \( \{ \text{2 bin. searches!} \) \( O(\log n) \) time!

**But:** Space? \( \Theta(n^2) \)
What’s the Problem?

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Query: \[ \begin{align*} &- \text{find right slab} \\ &- \text{search slab} \end{align*} \] \{ 2 \text{ bin. searches!} \} \text{ 2 bin. searches!} \]

But: Space? $\Theta(n^2)$ Task: Give lower-bound example!
What’s the Problem?

**Task:** Given a planar subdivision $S$ with $n$ segments, preprocess $S$ to allow for fast point location queries!

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Query:  
- find right slab \[ \{ \text{2 bin. searches!} \]  
- search slab \[ O(\log n) \] time!

**But:** Space? $\Theta(n^2)$ Pre-proc?
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Query: \begin{align*}
- & \text{find right slab} \\
- & \text{search slab}
\end{align*}

\begin{align*}
\{ & 2 \text{ bin. searches!} \\
\} & O(\log n) \text{ time!}
\end{align*}

**But:** Space? $\Theta(n^2)$ Pre-proc? $O(n^2 \log n)$
Decreasing the Complexity

**Observation:** The slab partition of $S$ is a *refinement* $S'$ of $S$ that consists of (possibly degenerate) trapezoids.
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**Task:** Find “good” refinement of \( S \) of low complexity!
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**Observation:** The slab partition of $S$ is a refinement $S'$ of $S$ that consists of (possibly degenerate) trapezoids.

**Task:** Find “good” refinement of $S$ of low complexity!

**Solution:** Trapezoidal map $T(S)$
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![Diagram of trapezoidal map](image-url)
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Assumption: $S$ is in general position, that is, no two vertices have the same $x$-coordinates.
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See CG: A&A Ch. 6.3
Notation

Definition: A side of a face of $\mathcal{T}(S)$ is a segment of maximum length contained in the boundary of the face.
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**Observation:** $S$ in gen. pos. $\Rightarrow$ each face $\Delta$ of $\mathcal{T}(S)$ has:
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Definition: A side of a face of $T(S)$ is a segment of maximum length contained in the boundary of the face.

Observation: $S$ in gen. pos. $\Rightarrow$ each face $\Delta$ of $T(S)$ has:
- one or two vertical sides
Notation

Definition: A side of a face of $\mathcal{T}(S)$ is a segment of maximum length contained in the boundary of the face.

Observation: $S$ in gen. pos. $\Rightarrow$ each face $\Delta$ of $\mathcal{T}(S)$ has:
- one or two vertical sides
- exactly 2 non-vertical sides
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**Left side:**
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Left side:
Complexity of $\mathcal{T}(S)$

**Observe:** A face $\Delta$ of $\mathcal{T}(S)$ is uniquely defined by $\text{top}(\Delta)$, $\text{bot}(\Delta)$, $\text{leftp}(\Delta)$, and $\text{rightp}(\Delta)$. 
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**Lemma.** $S$ planar subdivision in gen. pos., with $n$ segments $\Rightarrow \mathcal{T}(S)$ has $\leq \quad$ vtc and $\leq \quad$ trapezoids.
Complexity of $\mathcal{T}(S)$

**Observe:** A face $\Delta$ of $\mathcal{T}(S)$ is uniquely defined by $\text{top}(\Delta)$, $\text{bot}(\Delta)$, $\text{leftp}(\Delta)$, and $\text{rightp}(\Delta)$.

**Lemma.** $S$ planar subdivision in gen. pos., with $n$ segments $\Rightarrow \mathcal{T}(S)$ has $\leq 6n + 4$ vtc and $\leq 3n + 1$ trapezoids.

**Proof.** The vertices of $\mathcal{T}(S)$ are
– endpts of segments in $S$
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- endpts of vertical extensions
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- endpts of segments in $S$ $\leq 2n$
- endpts of vertical extensions $\leq 2 \cdot 2n$
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- endpts of segments in $S$ $\leq 2n$
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- vertices of $R$
Complexity of $\mathcal{T}(S)$

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**Proof.** The vertices of $\mathcal{T}(S)$ are
\[
\begin{align*}
\text{endpts of segments in } S & \leq 2n \\
\text{endpts of vertical extensions} & \leq 2 \cdot 2n \\
\text{vertices of } R & = 4
\end{align*}
\]}
Complexity of $\mathcal{T}(S)$

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- endpts of vertical extensions \leq 2 \cdot 2n
- vertices of $R$ \leq 6n + 4

\[
\leq 2n \\
\leq 2 \cdot 2n \\
\leq 6n + 4
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- vertices of $R$ $\leq 4$

\[
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\leq 2n & \leq 2 \cdot 2n \\
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- vertices of $R$ $\leq 4$

$\{\leq 6n + 4}$

Bound $\#\text{trapezoids}$ via Euler or directly (segments/\text{leftp}).
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**Observe:** A face $\Delta$ of $\mathcal{T}(S)$ is uniquely defined by top($\Delta$), bot($\Delta$), leftp($\Delta$), and rightp($\Delta$).

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**Proof.** The vertices of $\mathcal{T}(S)$ are

- endpts of segments in $S$ $\leq 2n$
- endpts of vertical extensions $\leq 2 \cdot 2n$ $\leq 6n + 4$
- vertices of $R$

Bound $\#$trapezoids via Euler or directly (segments/leftp).

**Approach:**
Complexity of $\mathcal{T}(S)$

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**Proof.** The vertices of $\mathcal{T}(S)$ are

- endpts of segments in $S$ $\leq 2n$
- endpts of vertical extensions $\leq 2 \cdot 2n$
- vertices of $R$ $\frac{4}{4}$

$\leq 6n + 4$

Bound $\#\text{trapezoids}$ via Euler or directly (segments/leftp).

**Approach:** Construct tapezoidal map $\mathcal{T}(S)$ and point-location data structure $\mathcal{D}(S)$ for $\mathcal{T}(S)$ incrementally!
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$$\left\{ \leq 6n + 4 \right\}$$

Bound $\#\text{trapezoids}$ via Euler or directly (segments/leftp).

**Approach:** Construct tapezoidal map $\mathcal{T}(S)$ and point-location data structure $\mathcal{D}(S)$ for $\mathcal{T}(S)$ *incrementally*! algorithm-design paradigm!
The 1d-Problem

Given a set $S$ of $n$ real numbers...
The 1d-Problem

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Given a set $S$ of $n$ real numbers...

$S_{i-1} := \{s_1, \ldots, s_{i-1}\}$, \quad $I_{i-1} := \text{set of conn. comp. of } \mathbb{R} \setminus S_{i-1}$
The 1d-Problem

Given a set $S$ of $n$ real numbers...

$S_{i-1} := \{s_1, \ldots, s_{i-1}\}$, \quad $I_{i-1} := \text{set of conn. comp. of } \mathbb{R} \setminus S_{i-1}$

- pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$
The 1d-Problem

Given a set $S$ of $n$ real numbers...

$i \in \{1, \ldots, n\}$

- pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$

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Given a set $S$ of $n$ real numbers...

$S_{i-1} := \{s_1, \ldots, s_{i-1}\}$, $I_{i-1} :=$ set of conn. comp. of $\mathbb{R} \setminus S_{i-1}$

– pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$
– locate $s_i$ in the search structure $D_{i-1}$ of $S_{i-1}$
The 1d-Problem

Given a set $S$ of $n$ real numbers...

$S_{i-1} := \{s_1, \ldots, s_{i-1}\}$, \quad $I_{i-1} := \text{set of conn. comp. of } \mathbb{R} \setminus S_{i-1}$

- pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$
- locate $s_i$ in the search structure $\mathcal{D}_{i-1}$ of $S_{i-1}$
The 1d-Problem

Given a set $S$ of $n$ real numbers...

- pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$
- locate $s_i$ in the search structure $D_{i-1}$ of $S_{i-1}$
- split interval $(\ell, r)$ of $I_{i-1}$ containing $s_i$

$$S_{i-1} := \{s_1, \ldots, s_{i-1}\}, \quad I_{i-1} := \text{set of conn. comp. of } \mathbb{R} \setminus S_{i-1}$$

$i \in \{1, \ldots, n\}$
The 1d-Problem

Given a set $S$ of $n$ real numbers...

- pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$
- locate $s_i$ in the search structure $D_{i-1}$ of $S_{i-1}$
- split interval $(\ell, r)$ of $I_{i-1}$ containing $s_i$
- build $D_i$:

$$S_{i-1} := \{s_1, \ldots, s_{i-1}\}, \quad I_{i-1} := \text{set of conn. comp. of } \mathbb{R} \setminus S_{i-1}$$
The 1d-Problem

Given a set $S$ of $n$ real numbers...

- pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$
- locate $s_i$ in the search structure $D_{i-1}$ of $S_{i-1}$
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Given a set $S$ of $n$ real numbers...

- pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$
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\[ S_{i-1} := \{s_1, \ldots, s_{i-1}\}, \quad I_{i-1} := \text{set of conn. comp. of } \mathbb{R} \setminus S_{i-1}, \]

\[ i \in \{1, \ldots, n\} \]
The 1d-Problem

Given a set $S$ of $n$ real numbers...

$S_{i-1} := \{s_1, \ldots, s_{i-1}\}$, $I_{i-1} := \text{set of conn. comp. of } \mathbb{R} \setminus S_{i-1}$

- pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$
- locate $s_i$ in the search structure $D_{i-1}$ of $S_{i-1}$
- split interval $(\ell, r)$ of $I_{i-1}$ containing $s_i$
- build $D_i$: 

![Diagram showing the construction of $D_i$ from $D_{i-1}$]

$D_{i-1}$

$D_i$
The 1d-Problem

Given a set $S$ of $n$ real numbers...

$S_{i-1} := \{s_1, \ldots, s_{i-1}\}$, \hspace{1cm} $I_{i-1} := \text{set of conn. comp. of } \mathbb{R} \setminus S_{i-1}$

- pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$
- locate $s_i$ in the search structure $\mathcal{D}_{i-1}$ of $S_{i-1}$
- split interval $(\ell, r)$ of $I_{i-1}$ containing $s_i$
- build $\mathcal{D}_i$:

Problem:
The 1d-Problem

Given a set $S$ of $n$ real numbers...

$S_{i-1} := \{s_1, \ldots, s_{i-1}\}, \quad I_{i-1} := \text{set of conn. comp. of } \mathbb{R} \setminus S_{i-1}$

- pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$
- locate $s_i$ in the search structure $D_{i-1}$ of $S_{i-1}$
- split interval $(\ell, r)$ of $I_{i-1}$ containing $s_i$
- build $D_i$:

**Problem:** looong search paths!
The 1d-Problem

Given a set $S$ of $n$ real numbers...

$S_{i-1} := \{s_1, \ldots, s_{i-1}\}, \quad I_{i-1} := \text{set of conn. comp. of } \mathbb{R} \setminus S_{i-1}$

Solution:
- pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$
- locate $s_i$ in the search structure $D_{i-1}$ of $S_{i-1}$
- split interval $(\ell, r)$ of $I_{i-1}$ containing $s_i$
- build $D_i$:

Problem: looong search paths!
The 1d-Problem

Given a set $S$ of $n$ real numbers... $i \in \{1, \ldots, n\}$

$S_{i-1} := \{s_1, \ldots, s_{i-1}\}$, $I_{i-1} := \text{set of conn. comp. of } \mathbb{R} \setminus S_{i-1}$

Solution:
- pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$
- locate $s_i$ in the search structure $D_{i-1}$ of $S_{i-1}$
- split interval $(\ell, r)$ of $I_{i-1}$ containing $s_i$
- build $D_i$:

Problem: looong search paths!
The 1d-Problem

Given a set $S$ of $n$ real numbers...

$S_{i-1} := \{s_1, \ldots, s_{i-1}\}$, $I_{i-1} := \text{set of conn. comp. of } \mathbb{R} \setminus S_{i-1}$

**Solution:** random!

- pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$
- locate $s_i$ in the search structure $D_{i-1}$ of $S_{i-1}$
- split interval $(\ell, r)$ of $I_{i-1}$ containing $s_i$
- build $D_i$:

Problem: looong search paths!
The 1d-Problem

Given a set $S$ of $n$ real numbers...

- pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$
- locate $s_i$ in the search structure $D_{i-1}$ of $S_{i-1}$
- split interval $(\ell, r)$ of $I_{i-1}$ containing $s_i$
- build $D_i$:

**Solution:** random!
- pick an arbitrary point $s_i$ from $S \setminus S_{i-1}$
- locate $s_i$ in the search structure $D_{i-1}$ of $S_{i-1}$
- split interval $(\ell, r)$ of $I_{i-1}$ containing $s_i$
- build $D_i$:

**Problem:** looong search paths!
1d Result

Given a set $S$ of $n$ real numbers...

\[ S_{i-1} := \{s_1, \ldots, s_{i-1}\}, \quad I_{i-1} := \text{set of conn. comp. of } \mathbb{R} \setminus S_{i-1} \]

**Thm.** The randomized-incremental algorithm preprocesses a set $S$ of $n$ reals in $O(n \log n)$ expected time such that a query takes $O(\log n)$ expected time.
1d Result

Given a set $S$ of $n$ real numbers...

Let $q \in \mathbb{R}$ (wlog. $q \notin S$) and $l_i(q) = \arg\{l \in l_i : q \in l\}$.

**Thm.** The randomized-incremental algorithm preprocesses a set $S$ of $n$ reals in $O(n \log n)$ expected time such that a query takes $O(\log n)$ expected time.

**Proof.** Let $q \in \mathbb{R}$ (wlog. $q \notin S$) and $l_i(q) = \arg\{l \in l_i : q \in l\}$. 
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$E[\text{query time in } D_n] =$
1d Result

Given a set $S$ of $n$ real numbers...

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\[ E[\text{query time in } D_n] = E[\text{length search path in } D_n] = \]
1d Result

Given a set $S$ of $n$ real numbers...

The randomized-incremental algorithm preprocesses a set $S$ of $n$ reals in $O(n \log n)$ expected time such that a query takes $O(\log n)$ expected time.

**Thm.** The randomized-incremental algorithm preprocesses a set $S$ of $n$ reals in $O(n \log n)$ expected time such that a query takes $O(\log n)$ expected time.

**Proof.** Let $q \in \mathbb{R}$ (wlog. $q \not\in S$) and $l_i(q) = \text{arg}\{l \in l_i : q \in l\}$.

Define random variable $X_i = \begin{cases} 1 & \text{if } l_i(q) \neq l_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

$E[\text{query time in } D_n] = E[\text{length search path in } D_n] =$
1d Result

Given a set $S$ of $n$ real numbers...

The randomized-incremental algorithm preprocesses a set $S$ of $n$ reals in $O(n \log n)$ expected time such that a query takes $O(\log n)$ expected time.

**Thm.** The randomized-incremental algorithm preprocesses a set $S$ of $n$ reals in $O(n \log n)$ expected time such that a query takes $O(\log n)$ expected time.

**Proof.** Let $q \in \mathbb{R}$ (wlog. $q \notin S$) and $l_i(q) = \arg\{l \in l_i : q \in l\}$.

Define random variable $X_i = \begin{cases} 1 & \text{if } l_i(q) \neq l_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

$E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] = E[\sum_{i=1}^n X_i] =$
1d Result

Given a set $S$ of $n$ real numbers...

$$S_{i-1} := \{s_1, \ldots, s_{i-1}\}, \quad l_{i-1} := \text{set of conn. comp. of } \mathbb{R} \setminus S_{i-1}$$

**Thm.** The randomized-incremental algorithm preprocesses a set $S$ of $n$ reals in $O(n \log n)$ expected time such that a query takes $O(\log n)$ expected time.

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$$E[\text{query time in } D_n] = E[\text{length search path in } D_n] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = ?$$
Expected Query Time of $\mathcal{D}_n$

Define random variable $X_i = \begin{cases} 
1 & \text{if } l_i(q) \neq l_{i-1}(q), \\
0 & \text{else.} 
\end{cases}$

$E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] = 
E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = ?$
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$E[X_i] = \begin{cases} 
1 & \text{if } l_i(q) \neq l_{i-1}(q), \\
0 & \text{else.}
\end{cases}$

$E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] =$

$= E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = ?$
Expected Query Time of $\mathcal{D}_n$

\[ E[X_i] = P[X_i = 1] = \]

Define random variable $X_i = \begin{cases} 
1 & \text{if } l_i(q) \neq l_{i-1}(q), \\
0 & \text{else.} 
\end{cases} 

E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] = 
\sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} E[X_i] = ?
Expected Query Time of $\mathcal{D}_n$

\[
E[X_i] = P[X_i = 1] =
= \text{probability that } l_i(q) \neq l_{i-1}(q)
\]

Define random variable $X_i = \begin{cases} 1 & \text{if } l_i(q) \neq l_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

\[
E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] =
= E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = ?
\]
Expected Query Time of $\mathcal{D}_n$

$$E[X_i] = P[X_i = 1] =$$

= probability that $l_i(q) \neq l_{i-1}(q)$, i.e., $s_i \in l_{i-1}(q)$.

Define random variable $X_i = \begin{cases} 
1 & \text{if } l_i(q) \neq l_{i-1}(q), \\
0 & \text{else.}
\end{cases}$

$$E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] =$$

$$= E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i] = ?$$
Expected Query Time of $D_n$

\[ E[X_i] = P[X_i = 1] = \]
\[ = \text{probability that } l_i(q) \neq l_{i-1}(q), \text{ i.e., } s_i \in l_{i-1}(q). \]

**Backwards analysis:**

Define random variable $X_i = \begin{cases} 1 & \text{if } l_i(q) \neq l_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

\[ E[\text{query time in } D_n] = E[\text{length search path in } D_n] = \]
\[ = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = ? \]
Expected Query Time of $\mathcal{D}_n$

$$E[X_i] = P[X_i = 1] =$$

= probability that $l_i(q) \neq l_{i-1}(q)$, i.e., $s_i \in l_{i-1}(q)$.

Backwards analysis: Consider $S_i$ fixed.

Define random variable $X_i = \begin{cases} 1 & \text{if } l_i(q) \neq l_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

$$E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] =$$

$$= E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = ?$$
Expected Query Time of $\mathcal{D}_n$

$$E[X_i] = P[X_i = 1] =$$
$$= \text{probability that } I_i(q) \neq I_{i-1}(q), \text{ i.e., } s_i \in I_{i-1}(q).$$

**Backwards analysis:** Consider $S_i$ fixed.

If we remove a randomly chosen pt from $S_i$,

Define random variable $X_i = \begin{cases} 1 & \text{if } I_i(q) \neq I_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

$$E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] =$$
$$= E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i] = ?$$
Expected Query Time of $\mathcal{D}_n$

$$E[X_i] = P[X_i = 1] =$$

= probability that $l_i(q) \neq l_{i-1}(q)$, i.e., $s_i \in l_{i-1}(q)$.

**Backwards analysis:**

Consider $S_i$ fixed.

If we remove a randomly chosen pt from $S_i$, what’s the probability that the interval containing $q$ changes?

Define random variable $X_i = \begin{cases} 1 & \text{if } l_i(q) \neq l_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

$$E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] =$$

$$= E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = ?$$
Expected Query Time of $\mathcal{D}_n$

$$E[X_i] = P[X_i = 1] =$$

$$= \text{probability that } l_i(q) \neq l_{i-1}(q), \text{ i.e., } s_i \in l_{i-1}(q).$$

**Backwards analysis:** Consider $S_i$ fixed.

If we *remove* a randomly chosen pt from $S_i$, what’s the probability that the interval containing $q$ changes?

– we have $i$ choices, identically distributed

Define random variable $X_i = \begin{cases} 1 & \text{if } l_i(q) \neq l_{i-1}(q), \\
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$$E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] =$$

$$= E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = ?$$
Expected Query Time of $\mathcal{D}_n$

$$E[X_i] = P[X_i = 1] = \text{probability that } l_i(q) \neq l_{i-1}(q), \text{ i.e., } s_i \in l_{i-1}(q).$$

**Backwards analysis:** Consider $S_i$ fixed.

If we *remove* a randomly chosen pt from $S_i$, what’s the probability that the interval containing $q$ changes?

– we have $i$ choices, identically distributed

– at most two of these change the interval

Define random variable $X_i = \begin{cases} 1 & \text{if } l_i(q) \neq l_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

$$E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = ?$$
Expected Query Time of $\mathcal{D}_n$

$$E[X_i] = P[X_i = 1] = \text{probability that } l_i(q) \neq l_{i-1}(q), \text{ i.e., } s_i \in l_{i-1}(q).$$

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$$E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = ?$$
**Expected Query Time of** $\mathcal{D}_n$

\[
E[X_i] = P[X_i = 1] = \frac{2}{i}
\]

= probability that $l_i(q) \neq l_{i-1}(q)$, i.e., $s_i \in l_{i-1}(q)$.

**Backwards analysis:** Consider $S_i$ fixed.

If we remove a randomly chosen pt from $S_i$, what’s the probability that the interval containing $q$ changes?

– we have $i$ choices, identically distributed

– at most two of these change the interval

Define random variable $X_i = \begin{cases} 1 & \text{if } l_i(q) \neq l_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

\[
E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] =
\]

\[
= E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = ?
\]
Expected Query Time of $\mathcal{D}_n$

Define random variable $X_i = \begin{cases} 1 & \text{if } l_i(q) \neq l_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

$E[X_i] = P[X_i = 1] = \frac{2}{i}$

= probability that $l_i(q) \neq l_{i-1}(q)$, i.e., $s_i \in l_{i-1}(q)$.

Backwards analysis: Consider $S_i$ fixed.

If we remove a randomly chosen pt from $S_i$, what's the probability that the interval containing $q$ changes?
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Define random variable $X_i = \begin{cases} 1 & \text{if } l_i(q) \neq l_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

$E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = ?$
Expected Query Time of $D_n$

\[ E[X_i] = P[X_i = 1] = \frac{2}{i} \]

= probability that $l_i(q) \neq l_{i-1}(q)$, i.e., $s_i \in l_{i-1}(q)$.

**Backwards analysis:** Consider $S_i$ fixed.  

If we remove a randomly chosen pt from $S_i$, what’s the probability that the interval containing $q$ changes? 
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Define random variable $X_i = \begin{cases} 1 & \text{if } l_i(q) \neq l_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

\[ E[\text{query time in } D_n] = E[\text{length search path in } D_n] = \]

\[ = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = ? \]

$O(\log n)$
The randomized-incremental algorithm preprocesses a set $S$ of $n$ reals in $O(n \log n)$ expected time such that a query takes $O(\log n)$ expected time.
The 2d-Problem

**Approach:** randomized-incremental construction of $T$ and $D$
The 2d-Problem

**Approach:** randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$
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point-location data structure (DAG)
trapezoidal map
The 2d-Problem

**Approach:** randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

– use $\mathcal{D}$ to locate left endpoint of next segment $s$
The 2d-Problem

**Approach:** randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$
- use $\mathcal{D}$ to locate left endpoint of next segment $s$
- “walk” along $s$ through $\mathcal{T}$
The 2d-Problem

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$
- use $\mathcal{D}$ to locate left endpoint of next segment $s$
- “walk” along $s$ through $\mathcal{T}$
- destroy all trapezoids of $\mathcal{T}$ intersecting $s$
The 2d-Problem

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$
- use $\mathcal{D}$ to locate left endpoint of next segment $s$
- “walk” along $s$ through $\mathcal{T}$
- destroy all trapezoids of $\mathcal{T}$ intersecting $s$
- construct new trapezoids of $\mathcal{T}$ (adjacent to $s$)

point-location data structure (DAG)
trapezoidal map
The 2d-Problem

**Approach:** randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$
- use $\mathcal{D}$ to locate left endpoint of next segment $s$
- “walk” along $s$ through $\mathcal{T}$
- destroy all trapezoids of $\mathcal{T}$ intersecting $s$
- construct new trapezoids of $\mathcal{T}$ (adjacent to $s$)
- update $\mathcal{D}$
Walking through $\mathcal{T}$ and Updating $\mathcal{D}$

TrapezoidalMap(set $S$ of $n$ non-crossing segments)

$$R = \text{BBox}(S); \quad \mathcal{T}.\text{init}(); \quad \mathcal{D}.\text{init}()$$

$$(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$$

\textbf{for} $i = 1$ \textbf{to} $n$ \textbf{do}

\begin{align*}
\Delta_0 &\quad \Delta_1 \\
\Delta_2 &\quad \Delta_3
\end{align*}
Walking through $\mathcal{T}$ and Updating $\mathcal{D}$

\[
\Delta_0 \quad \Delta_1 \quad \Delta_2 \quad \Delta_3
\]

\[
\mathcal{T}(S_{i-1})
\]

TrapezoidalMap(set $S$ of $n$ non-crossing segments)

\[
R = \text{BBox}(S); \quad \mathcal{T}.\text{init}(); \quad \mathcal{D}.\text{init}()
\]

\[
(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)
\]

\[
\text{for } i = 1 \text{ to } n \text{ do}
\]

\[
(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)
\]

\[
\mathcal{D}.\text{add new inner nodes()}
\]

\[
\mathcal{D}.\text{add leaves(new trapezoids incident to } s_i)
\]

\[
\mathcal{D}.\text{remove leaves(}\Delta_0, \ldots, \Delta_k\text{)}
\]

\[
\mathcal{T}.\text{remove(}\Delta_0, \ldots, \Delta_k\text{)}
\]

\[
\mathcal{T}.\text{add(new trapezoids incident to } s_i)
\]
Walking through $T$ and Updating $D$

TrapezoidalMap(set $S$ of $n$ non-crossing segments)

$$R = \text{BBox}(S); \ T.\text{init}(); \ D.\text{init}()$$

$$(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$$

for $i = 1$ to $n$ do

$$(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(T, D, s_i)$$

$$\ T.\text{remove}(\Delta_0, \ldots, \Delta_k)$$

$$\ T.\text{add(new trapezoids incident to } s_i)$$

$$\ D.\text{remove leaves}(\Delta_0, \ldots, \Delta_k)$$

$$\ D.\text{add leaves(new trapezoids incident to } s_i)$$

$$\ D.\text{add new inner nodes}()$$

$$R = \text{BBox}(S); \ T.\text{init}(); \ D.\text{init}()$$

$$(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$$

for $i = 1$ to $n$ do

$$(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(T, D, s_i)$$

$$\ T.\text{remove}(\Delta_0, \ldots, \Delta_k)$$

$$\ T.\text{add(new trapezoids incident to } s_i)$$

$$\ D.\text{remove leaves}(\Delta_0, \ldots, \Delta_k)$$

$$\ D.\text{add leaves(new trapezoids incident to } s_i)$$

$$\ D.\text{add new inner nodes}()$$
Walking through $\mathcal{T}$ and Updating $\mathcal{D}$

\[
\begin{align*}
\text{TrapezoidalMap(set } S \text{ of } n \text{ non-crossing segments)} \\
R &= \text{BBox}(S); \mathcal{T}.\text{init}(); \mathcal{D}.\text{init}() \\
(s_1, s_2, \ldots, s_n) &= \text{RandomPermutation}(S) \\
\text{for } i = 1 \text{ to } n \text{ do} \\
\quad (\Delta_0, \ldots, \Delta_k) &= \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i) \\
\quad \mathcal{T}.\text{remove}(\Delta_0, \ldots, \Delta_k) \\
\quad \mathcal{D}.\text{remove}\text{leaves}(\Delta_0, \ldots, \Delta_k) \\
\quad \mathcal{D}.\text{add}\text{leaves}(\text{new trapezoids incident to } s_i) \\
\quad \mathcal{D}.\text{add}\text{new inner nodes()}
\end{align*}
\]
Walking through $\mathcal{T}$ and Updating $\mathcal{D}$

TrapezoidalMap(set $S$ of $n$ non-crossing segments)

$R = \text{BBox}(S); \mathcal{T}.\text{init}(); \mathcal{D}.\text{init}()$

$(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$

for $i = 1$ to $n$ do

$(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$

$\mathcal{T}.\text{remove}(\Delta_0, \ldots, \Delta_k)$

$\mathcal{D}.\text{remove leaves}(\Delta_0, \ldots, \Delta_k)$

$\mathcal{D}.\text{add leaves}(\text{new trapezoids incident to } s_i)$

$\mathcal{D}.\text{add new inner nodes}()$
Walking through $\mathcal{T}$ and Updating $\mathcal{D}$

TrapezoidalMap(set $S$ of $n$ non-crossing segments)

$R = \text{bbox}(S); \mathcal{T}.\text{init}(); \mathcal{D}.\text{init}()$

$(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$

for $i = 1$ to $n$ do

$(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$

$\mathcal{T}.\text{remove}(\Delta_0, \ldots, \Delta_k)$

$\mathcal{T}.\text{add}(\text{new trapezoids incident to } s_i)$

)
Walking through $T$ and Updating $D$

\[
\begin{align*}
T(S_{i-1}) & \quad T(S_i) \\
\Delta_0 & \quad \Delta_0 \\
\Delta_1 & \quad \Delta_1 \\
\Delta_2 & \quad \Delta_2 \\
\Delta_3 & \quad \Delta_3 \\
p_i & \quad A \\
s_i & \quad B \\
q_i & \quad C \\
\end{align*}
\]

TrapezoidalMap(set $S$ of $n$ non-crossing segments)

\[
\begin{align*}
R &= \text{BBox}(S); T.\text{init}(); D.\text{init}() \\
(s_1, s_2, \ldots, s_n) &= \text{RandomPermutation}(S) \\
\text{for } i = 1 \text{ to } n \text{ do } \\
(\Delta_0, \ldots, \Delta_k) &= \text{FollowSegment}(T, D, s_i) \\
T.\text{remove}(\Delta_0, \ldots, \Delta_k) \\
T.\text{add}(\text{new trapezoids incident to } s_i) \\
D.\text{remove leaves}(\Delta_0, \ldots, \Delta_k) \\
D.\text{add leaves}(\text{new trapezoids incident to } s_i) \\
D.\text{add new inner nodes()}
\end{align*}
\]
Walking through $\mathcal{T}$ and Updating $\mathcal{D}$

\textbf{TrapezoidalMap(set $S$ of $n$ non-crossing segments)}

\[ R = \text{BBox}(S); \; \mathcal{T}.\text{init}(); \; \mathcal{D}.\text{init}() \]

\[(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S) \]

\textbf{for $i = 1$ to $n$ do}

\[ (\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i) \]

\[ \mathcal{T}.\text{remove}(\Delta_0, \ldots, \Delta_k) \]

\[ \mathcal{T}.\text{add(new trapezoids incident to } s_i) \]
Walking through $\mathcal{T}$ and Updating $\mathcal{D}$

TrapezoidalMap(set $S$ of $n$ non-crossing segments)

$$R = \text{BBox}(S); \quad \mathcal{T}.\text{init}(); \quad \mathcal{D}.\text{init}()$$

$$(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$$

for $i = 1$ to $n$ do
\begin{align*}
(\Delta_0, \ldots, \Delta_k) &= \text{FollowSegment}(& \mathcal{T}, \mathcal{D}, s_i) \\
\mathcal{T}.\text{remove}(\Delta_0, \ldots, \Delta_k) \\
\mathcal{T}.\text{add(new trapezoids incident to } s_i) \\
\mathcal{D}.\text{remove_leaves}(\Delta_0, \ldots, \Delta_k)
\end{align*}

)
Walking through $T$ and Updating $D$

TrapezoidalMap(set $S$ of $n$ non-crossing segments)

$R = \text{BBox}(S); T.\text{init}(); D.\text{init}()$

$(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$

for $i = 1$ to $n$ do

$(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(T, D, s_i)$

$T.\text{remove}(\Delta_0, \ldots, \Delta_k)$

$T.\text{add}(\text{new trapezoids incident to } s_i)$

$D.\text{remove leaves}(\Delta_0, \ldots, \Delta_k)$
Walking through $\mathcal{T}$ and Updating $\mathcal{D}$

TrapezoidalMap(set $S$ of $n$ non-crossing segments)

$R = \text{BBox}(S)$; $\mathcal{T}$.init(); $\mathcal{D}$.init()

$(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$

for $i = 1$ to $n$ do

$(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$

$\mathcal{T}$.remove($\Delta_0, \ldots, \Delta_k$)

$\mathcal{T}$.add(new trapezoids incident to $s_i$)

$\mathcal{D}$.remove_leaves($\Delta_0, \ldots, \Delta_k$)

$\mathcal{D}$.add_leaves(new trapezoids incident to $s_i$)

$\mathcal{D}$.add_new_inner_nodes()
Walking through $\mathcal{T}$ and Updating $\mathcal{D}$

TrapezoidalMap(set $S$ of $n$ non-crossing segments)

$R = BBox(S); \mathcal{T}.\text{init}(); \mathcal{D}.\text{init}()$

$(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$

for $i = 1$ to $n$ do

$(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$

$\mathcal{T}.\text{remove}(\Delta_0, \ldots, \Delta_k)$

$\mathcal{T}.\text{add}(\text{new trapezoids incident to $s_i$})$

$\mathcal{D}.\text{remove_leaves}(\Delta_0, \ldots, \Delta_k)$

$\mathcal{D}.\text{add_leaves}(\text{new trapezoids incident to $s_i$})$
Walking through $\mathcal{T}$ and Updating $\mathcal{D}$

TrapezoidalMap(set $S$ of $n$ non-crossing segments)

$R = \text{BBox}(S); \mathcal{T}.\text{init}(); \mathcal{D}.\text{init}()$

$(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$

for $i = 1$ to $n$ do

$(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$

$\mathcal{T}.\text{remove}(\Delta_0, \ldots, \Delta_k)$

$\mathcal{T}.\text{add(new trapezoids incident to } s_i)$

$\mathcal{D}.\text{remove leaves}(\Delta_0, \ldots, \Delta_k)$

$\mathcal{D}.\text{add leaves(new trapezoids incident to } s_i)$

$\mathcal{D}.\text{add new inner nodes()}$
Walking through $\mathcal{T}$ and Updating $\mathcal{D}$

TrapezoidalMap(set $S$ of $n$ non-crossing segments)

$R = \text{BBox}(S); \mathcal{T}.\text{init}(); \mathcal{D}.\text{init}()$

$(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$

for $i = 1$ to $n$ do

$(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$

$\mathcal{T}.\text{remove}((\Delta_0, \ldots, \Delta_k))$

$\mathcal{T}.\text{add}(\text{new trapezoids incident to } s_i)$

$\mathcal{D}.\text{remove\_leaves}((\Delta_0, \ldots, \Delta_k))$

$\mathcal{D}.\text{add\_leaves}(\text{new trapezoids incident to } s_i)$

$\mathcal{D}.\text{add\_new\_inner\_nodes}()$
Theorem. TrapezoidalMap(S) computes $\mathcal{T}(S)$ for a set of $n$ line segments in general position and a search structure $\mathcal{D}$ for $\mathcal{T}(S)$ in $O(n \log n)$ expected time.
Theorem. TrapezoidalMap(S) computes \( T(S) \) for a set of \( n \) line segments in general position and a search structure \( D \) for \( T(S) \) in \( O(n \log n) \) expected time. The expected size of \( D \) is \( O(n) \) and the expected query time is \( O(\log n) \).
The 2d-Result

**Theorem.** TrapezoidalMap($S$) computes $T(S)$ for a set of $n$ line segments in general position and a search structure $D$ for $T(S)$ in $O(n \log n)$ expected time. The expected size of $D$ is $O(n)$ and the expected query time is $O(\log n)$.

**Invariant:** Before step $i$, $T$ is a trapezoidal map for $S_{i-1}$ and $D$ is a valid search structure for $T$.

**Proof.**
- Correctness by loop invariant.
- Query time similar to 1d analysis.
  $\Rightarrow$ construction time
Query Time

Let $T(q)$ be the query time for a fixed query pt $q$. 

Query Time

Let $T(q)$ be the query time for a fixed query pt $q$.

$\Rightarrow T(q) = O(\quad )$. 
Query Time

Let $T(q)$ be the query time for a fixed query pt $q$.

$\Rightarrow T(q) = O(\text{length of the path from } D\text{.root to } q)$. 
Query Time

Let $T(q)$ be the query time for a fixed query pt $q$.

$\Rightarrow T(q) = O($length of the path from $D$.root to $q$).

height($D$) increases by at most 3 in each step.
Query Time

Let $T(q)$ be the query time for a fixed query pt $q$. 

$\Rightarrow T(q) = O(\text{length of the path from } D\text{.root to } q)$. 

height($D$) increases by at most 3 in each step. $\Rightarrow T(q) \leq$
Query Time

Let $T(q)$ be the query time for a fixed query pt $q$.

$\Rightarrow T(q) = O(\text{length of the path from } D.\text{root to } q)$.

height($D$) increases by at most 3 in each step. $\Rightarrow T(q) \leq 3n$. 
Query Time

Let $T(q)$ be the query time for a fixed query pt $q$.

$\Rightarrow T(q) = O(\text{length of the path from } D\text{.root to } q)$.

height($D$) increases by at most 3 in each step.  $\Rightarrow T(q) \leq 3n$.

We are interested in the expected behaviour of $D$: 
Query Time

Let $T(q)$ be the query time for a fixed query pt $q$.

$\Rightarrow T(q) = O(\text{length of the path from } D.\text{root to } q)$. 

height($D$) increases by at most 3 in each step.  $\Rightarrow T(q) \leq 3n$.

We are interested in the expected behaviour of $D$:

$\Rightarrow$ average of $T(q)$ over $3n$. 
Query Time

Let $T(q)$ be the query time for a fixed query pt $q$. 
$\Rightarrow T(q) = O(\text{length of the path from } D.\text{root to } q)$. 

height($D$) increases by at most 3 in each step. $\Rightarrow T(q) \leq 3n$.

We are interested in the expected behaviour of $D$: 
$\Rightarrow$ average of $T(q)$ over all $n!$ insertion orders
Query Time

Let $T(q)$ be the query time for a fixed query pt $q$.

$\Rightarrow T(q) = O(\text{length of the path from } D.\text{root to } q)$.

height($D$) increases by at most 3 in each step.  $\Rightarrow T(q) \leq 3n$.

We are interested in the expected behaviour of $D$:

$\Rightarrow$ average of $T(q)$ over all $n!$ insertion orders (permut. of $S$)
Query Time

Let $T(q)$ be the query time for a fixed query pt $q$.
$\Rightarrow T(q) = O(\text{length of the path from } D\text{.root to } q)$.

height($D$) increases by at most 3 in each step. $\Rightarrow T(q) \leq 3n$.

We are interested in the expected behaviour of $D$:
$\Rightarrow$ average of $T(q)$ over all $n!$ insertion orders (permut. of $S$)

$X_i := \# \text{ nodes that are added to the query path in iteration } i$. 
Query Time

Let $T(q)$ be the query time for a fixed query pt $q$.

$\Rightarrow T(q) = O(\text{length of the path from } D\text{.root to } q)$.

height($D$) increases by at most 3 in each step. $\Rightarrow T(q) \leq 3n$.

We are interested in the expected behaviour of $D$:

$\Rightarrow$ average of $T(q)$ over all $n!$ insertion orders (permut. of $S$)

$X_i := \# \text{ nodes that are added to the query path in iteration } i$.

$S$ and $q$ are fixed.
Query Time

Let $T(q)$ be the query time for a fixed query pt $q$.

$\Rightarrow T(q) = O(\text{length of the path from } \mathcal{D}.\text{root to } q)$.

height($\mathcal{D}$) increases by at most 3 in each step. $\Rightarrow T(q) \leq 3n$.

We are interested in the expected behaviour of $\mathcal{D}$:
$\Rightarrow$ average of $T(q)$ over all $n!$ insertion orders (permut. of $S$)

$X_i := \# \text{ nodes that are added to the query path in iteration } i$.

$S$ and $q$ are fixed.

$\Rightarrow X_i$ random variable that depends only on insertion order of $S$. 
Query Time

Let \( T(q) \) be the query time for a fixed query pt \( q \).
\[ \Rightarrow T(q) = O(\text{length of the path from } D\text{.root to } q). \]

height\((D)\) increases by at most 3 in each step. \( \Rightarrow T(q) \leq 3n. \)

We are interested in the expected behaviour of \( D \):
\[ \Rightarrow \text{average of } T(q) \text{ over all } n! \text{ insertion orders (permut. of } S) \]

\( X_i := \# \text{ nodes that are added to the query path in iteration } i. \)
\( S \) and \( q \) are fixed.
\[ \Rightarrow X_i \text{ random variable that depends only on insertion order of } S. \]
\[ \Rightarrow \text{expected path length from } D\text{.root to } q \text{ is} \]
Query Time

Let $T(q)$ be the query time for a fixed query pt $q$.

⇒ $T(q) = O(\text{length of the path from } D.\text{root to } q)$.

height($D$) increases by at most 3 in each step. ⇒ $T(q) \leq 3n$.

We are interested in the expected behaviour of $D$:

⇒ average of $T(q)$ over all $n!$ insertion orders (permut. of $S$)

$X_i := \# \text{ nodes that are added to the query path in iteration } i$. $S$ and $q$ are fixed.

⇒ $X_i$ random variable that depends only on insertion order of $S$.

⇒ expected path length from $D.\text{root to } q$ is

$$E[\sum_{i=1}^{n} X_i] =$$
Query Time

Let $T(q)$ be the query time for a fixed query pt $q$.
$\Rightarrow T(q) = O(\text{length of the path from } \mathcal{D}.\text{root to } q).$

height($\mathcal{D}$) increases by at most 3 in each step. $\Rightarrow T(q) \leq 3n.$

We are interested in the expected behaviour of $\mathcal{D}$:
$\Rightarrow$ average of $T(q)$ over all $n!$ insertion orders (permut. of $S$)

$X_i := \# \text{ nodes that are added to the query path in iteration } i.$

$S$ and $q$ are fixed.
$\Rightarrow X_i \text{ random variable that depends only on insertion order of } S.$
$\Rightarrow \text{ expected path length from } \mathcal{D}.\text{root to } q \text{ is}$

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = ?$$
Query Time (cont’d)

\[ p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } \mathcal{D} \text{ contains a node that was created in iteration } i. \]
Query Time (cont’d)

\[ p_i = \text{prob. that the search path } \pi_q \text{ of } q \text{ in } D \text{ contains a node that was created in iteration } i. \]

\[ \Rightarrow \mathbb{E}[X_i] = \]
Query Time (cont’d)

\[ p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } \mathcal{D} \text{ contains a node that was created in iteration } i. \]

\[ \Rightarrow \mathbb{E}[X_i] = \sum_{j=0}^{3} j \cdot P[X_i = j] \leq \]
Query Time (cont’d)

\( p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } D \text{ contains a node that was created in iteration } i. \)

\[ \Rightarrow E[X_i] = \sum_{j=0}^{3} j \cdot P[X_i = j] \leq \sum_{j=0}^{3} 3 \cdot P[X_i \geq 1] = \]
Query Time (cont’d)

\( p_i \) = prob. that the search path \( \Pi_q \) of \( q \) in \( D \) contains a node that was created in iteration \( i \).

\[ \Rightarrow E[X_i] = \sum_{j=0}^{3} j \cdot P[X_i = j] \leq \sum_{j=0}^{3} 3 \cdot P[X_i \geq 1] = 3p_i \]
Query Time (cont’d)

$p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } D \text{ contains a node that was created in iteration } i$.

$\Rightarrow E[X_i] = \sum_{j=0}^{3} j \cdot P[X_i = j] \leq \sum_{j=0}^{3} 3 \cdot P[X_i \geq 1] = 3p_i$

$\Delta_q(S_i) := \text{trapezoid in } T(S_i) \text{ that contains } q$. 
Query Time (cont’d)

\( p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } D \text{ contains a node that was created in iteration } i. \)

\[ \Rightarrow E[X_i] = \sum_{j=0}^{3} j \cdot P[X_i = j] \leq \sum_{j=0}^{3} 3 \cdot P[X_i \geq 1] = 3p_i \]

\( \Delta_q(S_i) := \text{trapezoid in } T(S_i) \text{ that contains } q. \)

**Key idea:** Iteration \( i \) contributes a node to \( \Pi_q \) iff
Query Time (cont’d)

\( p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } \mathcal{D} \text{ contains a node that was created in iteration } i. \)

\[ \Rightarrow \mathbb{E}[X_i] = \sum_{j=0}^{3} j \cdot P[X_i = j] \leq \sum_{j=0}^{3} 3 \cdot P[X_i \geq 1] = 3p_i \]

\( \Delta_q(S_i) := \text{trapezoid in } \mathcal{T}(S_i) \text{ that contains } q. \)

**Key idea:** Iteration \( i \) contributes a node to \( \Pi_q \) iff 

\( \Delta_q(S_{i-1}) \neq \Delta_q(S_i). \)
Query Time (cont’d)

\[ p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } D \text{ contains a node that was created in iteration } i. \]

\[ \Rightarrow E[X_i] = \sum_{j=0}^{3} j \cdot P[X_i = j] \leq \sum_{j=0}^{3} 3 \cdot P[X_i \geq 1] = 3p_i \]

\[ \Delta_q(S_i) := \text{trapezoid in } T(S_i) \text{ that contains } q. \]

**Key idea:** Iteration \( i \) contributes a node to \( \Pi_q \) iff

\[ \Delta_q(S_{i-1}) \neq \Delta_q(S_i). \]

In this case \( \Delta_q(S_i) \) must have been created in iteration \( i \).
Query Time (cont’d)

\( p_i \) = prob. that the search path \( \Pi_q \) of \( q \) in \( D \) contains a node that was created in iteration \( i \).

\[ \Rightarrow E[X_i] = \sum_{j=0}^{3} j \cdot P[X_i = j] \leq \sum_{j=0}^{3} 3 \cdot P[X_i \geq 1] = 3p_i \]

\( \Delta_q(S_i) := \) trapezoid in \( T(S_i) \) that contains \( q \).

**Key idea:** Iteration \( i \) contributes a node to \( \Pi_q \) iff

\[ \Delta_q(S_{i-1}) \neq \Delta_q(S_i). \]

In this case \( \Delta_q(S_i) \) must have been created in iteration \( i \).

\[ \Rightarrow \Delta := \Delta_q(S_i) \text{ is adjacent to the new segment } s_i. \]
Query Time (cont’d)

\( p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } D \text{ contains a node that was created in iteration } i. \)

\[ \Rightarrow E[X_i] = \sum_{j=0}^{3} j \cdot P[X_i = j] \leq \sum_{j=0}^{3} 3 \cdot P[X_i \geq 1] = 3p_i \]

\( \Delta_q(S_i) := \text{trapezoid in } T(S_i) \text{ that contains } q. \)

**Key idea:** Iteration \( i \) contributes a node to \( \Pi_q \) iff

\( \Delta_q(S_{i-1}) \neq \Delta_q(S_i). \)

In this case \( \Delta_q(S_i) \) must have been created in iteration \( i. \)

\( \Rightarrow \Delta := \Delta_q(S_i) \text{ is adjacent to the new segment } s_i. \)

\( \Rightarrow \text{top}(\Delta) = s_i, \text{bot}(\Delta) = s_i, \text{leftp}(\Delta) \in s_i, \text{ or rightp}(\Delta) \in s_i. \)
Query Time (cont’d)

\( p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } \mathcal{D} \text{ contains a node that was created in iteration } i. \)

\[ \Rightarrow E[X_i] = \sum_{j=0}^{3} j \cdot P[X_i = j] \leq \sum_{j=0}^{3} 3 \cdot P[X_i \geq 1] = 3p_i \]

\( \Delta_q(S_i) \) := trapezoid in \( \mathcal{T}(S_i) \) that contains \( q \).

\textbf{Key idea:} Iteration \( i \) contributes a node to \( \Pi_q \) iff \( \Delta_q(S_{i-1}) \neq \Delta_q(S_i) \).

In this case \( \Delta_q(S_i) \) must have been created in iteration \( i \).

\[ \Rightarrow \Delta := \Delta_q(S_i) \text{ is adjacent to the new segment } s_i. \]

\[ \Rightarrow \text{top}(\Delta) = s_i, \text{bot}(\Delta) = s_i, \text{leftp}(\Delta) \in s_i, \text{or rightp}(\Delta) \in s_i. \]

\textbf{Trick:} \( \mathcal{T}(S_i) \) (and thus \( \Delta \)) is uniquely determined by \( S_i \).
Query Time (cont’d)

\( p_i \) = prob. that the search path \( \Pi_q \) of \( q \) in \( D \) contains a node that was created in iteration \( i \).

\[ \Rightarrow E[X_i] = \sum_{j=0}^{3} j \cdot P[X_i = j] \leq \sum_{j=0}^{3} 3 \cdot P[X_i \geq 1] = 3p_i \]

\( \Delta_q(S_i) \) := trapezoid in \( T(S_i) \) that contains \( q \).

**Key idea:** Iteration \( i \) contributes a node to \( \Pi_q \) iff

\[ \Delta_q(S_{i-1}) \neq \Delta_q(S_i). \]

In this case \( \Delta_q(S_i) \) must have been created in iteration \( i \).

\[ \Rightarrow \Delta := \Delta_q(S_i) \text{ is adjacent to the new segment } s_i. \]

\[ \Rightarrow \text{top}(\Delta) = s_i, \text{bot}(\Delta) = s_i, \text{leftp}(\Delta) \in s_i, \text{or rightp}(\Delta) \in s_i. \]

**Trick:** \( T(S_i) \) (and thus \( \Delta \)) is uniquely determined by \( S_i \).

Consider \( S_i \subseteq S \) fixed.
Query Time (cont’d)

\( p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } \mathcal{D} \text{ contains a node that was created in iteration } i. \)

\[ \Rightarrow E[X_i] = \sum_{j=0}^{3} j \cdot P[X_i = j] \leq \sum_{j=0}^{3} 3 \cdot P[X_i \geq 1] = 3p_i \]

\( \Delta_q(S_i) := \text{trapezoid in } \mathcal{T}(S_i) \text{ that contains } q. \)

**Key idea:** Iteration \( i \) contributes a node to \( \Pi_q \) iff

\[ \Delta_q(S_{i-1}) \neq \Delta_q(S_i). \]

In this case \( \Delta_q(S_i) \) must have been created in iteration \( i. \)

\[ \Rightarrow \Delta := \Delta_q(S_i) \text{ is adjacent to the new segment } s_i. \]

\[ \Rightarrow \text{top}(\Delta) = s_i, \text{bot}(\Delta) = s_i, \text{leftp}(\Delta) \in s_i, \text{or rightp}(\Delta) \in s_i. \]

**Trick:** \( \mathcal{T}(S_i) \) (and thus \( \Delta \)) is uniquely determined by \( S_i. \)

Consider \( S_i \subseteq S \text{ fixed.} \)

\[ \Rightarrow \Delta \text{ does not depend on insertion order.} \]
Query Time (cont’d)

\( p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } \mathcal{D} \text{ contains a node that was created in iteration } i. \)

i.e., prob that \( \Delta \) changes when inserting \( s_i \).

**Aim:** bound \( p_i \).
Query Time (cont’d)

$p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } \mathcal{D} \text{ contains a node that was created in iteration } i.
\text{i.e., prob that } \Delta \text{ changes when inserting } s_i.$

**Aim:** bound $p_i.$

**Tool:**
Query Time (cont’d)

\[ p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } D \text{ contains a node that was created in iteration } i. \]

i.e., prob that \( \Delta \) changes when inserting \( s_i \).

**Aim:** bound \( p_i \).

**Tool:** *Backwards analysis!*
Aim: bound $p_i$.

Tool: Backwards analysis!

$p_i = \text{prob. that } \Delta \text{ changes when } s_i \text{ is removed}$
Query Time (cont’d)

$p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } D \text{ contains a node that was created in iteration } i.$

i.e., prob that $\Delta$ changes when inserting $s_i$.

\textbf{Aim:} bound $p_i$.

\textbf{Tool:} \textit{Backwards analysis!}

$p_i = \text{prob that } \Delta \text{ changes when } s_i \text{ is removed}$

Four cases:
Query Time (cont’d)

\[ p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } D \text{ contains a node that was created in iteration } i. \]

i.e., prob that \( \Delta \) changes when inserting \( s_i \).

**Aim:** bound \( p_i \).

**Tool:** *Backwards analysis!*

\[ p_i = \text{prob that } \Delta \text{ changes when removing } s_i. \]

Four cases:

\[ P(\text{top}(\Delta) = s_i) =? \]
Query Time (cont’d)

$p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } D \text{ contains a node that was created in iteration } i.$

i.e., prob that $\Delta$ changes when inserting $s_i$.

Aim: bound $p_i$.

Tool: Backwards analysis!

$p_i = \text{prob that } \Delta \text{ changes when } s_i \text{ is removed}$

Four cases:

$P(\text{top}(\Delta) = s_i) = 1/i$ (since exactly one of $i$ segments is $\text{top}(\Delta)$).
Query Time (cont’d)

$p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } D \text{ contains a node that was created in iteration } i.$

i.e., prob that $\Delta$ changes when inserting $s_i$.

**Aim:** bound $p_i$.

**Tool:** Backwards analysis!

$p_i = \text{prob that } \Delta \text{ changes when } s_i \text{ is removed}$

Four cases:

\[
P(\text{top}(\Delta) = s_i) = \frac{1}{i} \quad (\text{since exactly one of } i \text{ segments is } \text{top}(\Delta)).
\]

$\Rightarrow p_i \leq \frac{4}{i}$

$\Rightarrow E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] \leq \sum_{i=1}^{n} 3 \cdot p_i$

$= 12 \sum_{i=1}^{n} \frac{1}{i} \leq O(\log n)$


Query Time (cont’d)

\( p_i = \text{prob. that the search path } \Pi_q \text{ of } q \text{ in } \mathcal{D} \text{ contains a node that was created in iteration } i. \)

i.e., prob that \( \Delta \) changes when inserting \( s_i \).

**Aim:** bound \( p_i \).

**Tool:** Backwards analysis!

\( p_i = \text{prob that } \Delta \text{ changes when } s_i \text{ is removed} \)

Four cases:

\[
P(\text{top}(\Delta) = s_i) = \frac{1}{i} \quad \text{(since exactly one of } i \text{ segments is top}(\Delta)).
\]

\( \Rightarrow p_i \leq \frac{4}{i} \)

\( \Rightarrow \mathbb{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathbb{E}[X_i] \leq \sum_{i=1}^{n} 3 \cdot p_i \)

\( = 12 \sum_{i=1}^{n} \frac{1}{i} \leq O(\log n) \)