Algorithms for Graph Visualization

Summer Semester 2017
Lecture #10

The Crossing Lemma and its Applications

(based on the slides of Alexander Wolff and Philipp Kindermann)

DOI 10.1007/978-3-642-00856-6 (pp 256-258, proof of the crossing lemma)
Topological Graphs (figures from *Proofs from THE BOOK*)

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**Remark** Wlog, in a topological drawing, at most two edges intersect at the same point.
Crossings

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Only two options per edge: up/down
Rectilinear (straight-line) Crossing Number

For a graph $G$ the *rectilinear crossing number* of $G$ is:
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Separation: $\text{cr}(K_8) = 18$
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For a graph $G$ the *rectilinear crossing number* of $G$ is:

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**Obs.** For each $k \geq 4$ there is a graph $G_k$ with $\operatorname{cr}(G_k) = 4$ and $\operatorname{cr}(G_k) \geq k$. 
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\[ G_1 \]
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Each straight-line drawing of $G_1$ at least one crossing of the following types:

- [Diagram showing two types of crossings: $\cap$ or $\cup$.]
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$G_1 \rightarrow G_k$: 

![Diagram of $G_1$ and $G_k$]
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Each straight-line drawing of $G_1$ at least one crossing of the following types: $\overline{cr}(K_8) = 19$.
A first lower bound

**Obs**$_1$ A drawing of a graph $G$ with $n$ vertices and $m$ edges has at least $m - 3n + 6$ crossings.
A first lower bound

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\textbf{Proof.}  (blackboard)
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\textbf{Obs}_1 \quad \text{A drawing of a graph } G \text{ with } n \text{ vertices and } m \text{ edges has at least } m - 3n + 6 \text{ crossings.}

\textbf{Proof.} \quad (\text{blackboard})

\textbf{Obs}_2 \quad \text{Each drawing of } G \text{ has at least } r \cdot \left(\binom{\lfloor m/r \rfloor}{2}\right) \in \Omega(m^2/n) \text{ crossings, where } r \leq 3n - 6 \text{ is the maximum number of edges in a planar subgraph of } G.
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**Obs**\textsubscript{1} A drawing of a graph $G$ with $n$ vertices and $m$ edges has at least $m - 3n + 6$ crossings.

**Proof.** (blackboard)

**Obs**\textsubscript{2} Each drawing of $G$ has at least $r \cdot \left(\frac{\lfloor m/r \rfloor}{2}\right) \in \Omega(m^2/n)$ crossings, where $r \leq 3n - 6$ is the maximum number of edges in a planar subgraph of $G$.

**Proof** (blackboard)
Tighter Bounds

**Conj.** [Erdős & Guy ’73]

\[ \text{cr}(G) \in \Omega\left(\frac{m^3}{n^2}\right). \]
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**Thm** [Ajtai, Chvátal, Newborn, Szemerédi ’82, Leighton ’84]

\[ m \geq 4n \Rightarrow \text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}. \]

[Chazelle, Sharir, Welzl ...] “BOOK” proof (blackboard)
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**Remark** Bounds asymptotically sharp!
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** Remark** Bounds asymptotically sharp!

Consider geom. graph with vertices \( v_0, \ldots, v_{n-1} \) in convex position and \( E = \{ v_i v_j \mid i < j \leq i + k \mod n \} \) for \( 0 < k < n/2 \). [exercise!]
Tighter Bounds

**Conj.** [Erdős & Guy ’73]

\[ \text{cr}(G) \in \Omega \left( \frac{m^3}{n^2} \right). \]

**Thm** 1 [Ajtai, Chvátal, Newborn, Szemerédi ’82, Leighton ’84]

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“BOOK” proof (blackboard)

**Remark**

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**Thm** 2 [Pach & Tóth ’97]

Improving the constants

\[ m \geq 6n \Rightarrow \text{cr}(G) \geq \frac{1}{36} \cdot \frac{m^3}{n^2}. \]
Application 1: Point-Line Incidences

For points $P \subset \mathbb{R}^2$ and lines $\mathcal{L}$

$I(P, \mathcal{L}) =$ number of point-line incidences in $(P, \mathcal{L})$. 
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![Diagram of point-line incidences]
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Def. $I(n, k) = \max_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L})$.

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For example: $I(4, 4) =$
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Thm

$[\text{Szemerédi & Trotter '83, Székely '97}]

I(n, k) \leq 2.7\frac{n^2}{3}k^{2/3} + 6n + 2k.$

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Application 2: Unit Distances

For points $P \subset \mathbb{R}^2$ define
$U(P) =$ number of pairs in $P$ at unit distance
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**Thm** $4$ [Spencer, Szemerédi, Trotter ’84, Székely ’97]

$U(n) < 6.7n^{4/3}$