Algorithms for Graph Visualization

Summer Semester 2017

Lecture #9

Planar Orientations

Tessellations and Visibility Representations
Topological Numbering

Let $G = (V, E)$ be a directed graph.

- **Topological numbering** of $G$:
  mapping $\mu : V \rightarrow \mathbb{N}$ with $\mu(u) < \mu(v)$ for every edge $(u, v)$

- **Topological sort** of $G$:
  topological numbering where $\mu(V) = \{1, \ldots, n\}$

- **Weighted topological numbering** of $(G, w)$:
  topol. numb. with $\mu(u) + w(u, v) \leq \mu(v)$ for every edge $(u, v)$
  optimal when: $\max_{v \in V} \mu(v) - \min_{v \in V} \mu(v)$ is minimized.

- Can be calculated in $O(n + m)$ time. Exercise!
**st-graphs**

**st-graph**: a directed *acyclic* graph $G = (V, E)$ with exactly one source and exactly one sink.

$\quad \Rightarrow G$ numbered topologically: each path traverses nodes in increasing order.

$\quad \Rightarrow$ For any vertex $v$, there is a directed $(s, t)$-path containing $v$.

*Planar st-graph*: an st-graph with a planar embedding such that $s$ and $t$ are on the outer face.
Planar $st$-graphs

- Normally drawn upwards planar.
- Two outer faces $s^*/t^*$ left/right.
- For each $e = (u, v) \in E$:
  - $\text{orig}(e) = u$
  - $\text{dest}(e) = v$
  - $\text{left}(e) \in F$: face left of $e$
  - $\text{right}(e) \in F$: face right of $e$
- $G^* = (V^* = F, E^*)$:
  - $e \in E \Rightarrow (\text{left}(e), \text{right}(e)) \in E^*$
- Multigraph
- $s^*t^*$-graph
Properties of Planar $st$-Graphs

**Lemma 1** Every face $f$ of $G$ consists of two paths from its source $\text{orig}(f)$ to its sink $\text{dest}(f)$.

**Lemma 2** At each vertex $v \in V$, the incoming/outgoing edges each form an interval, and these intervals are separated by the faces $\text{left}(v)/\text{right}(v)$.

Statements imply the same in the dual.
Properties of Planar \textit{st}-graphs

\textbf{Lemma}_3 \quad \text{For faces } f \text{ and } g \text{ exactly one of the following is true:}
\begin{itemize}
  \item There is a path from \text{dest}(f) \text{ to } \text{orig}(g) \text{ in } G.
  \item There is a path from \text{dest}(g) \text{ to } \text{orig}(f) \text{ in } G.
  \item There is a path from \textit{f} \text{ to } \textit{g} \text{ in } G^*.
  \item There is a path from \textit{g} \text{ to } \textit{f} \text{ in } G^*.
\end{itemize}

For \( v \in V \): let \( \text{orig}(v) = \text{dest}(v) = v \);
For \( f \in F \): let \( \text{left}(f) = \text{right}(f) = f \).

\textbf{Lemma}_4 \quad \text{For objects } o_1, o_2 \in V \cup E \cup F \text{ exactly one of the following is true:}
\begin{itemize}
  \item There is a path from \text{dest}(o_1) \text{ to } \text{orig}(o_2) \text{ in } G.
  \item There is a path from \text{dest}(o_2) \text{ to } \text{orig}(o_1) \text{ in } G.
  \item There is a path from \text{right}(o_1) \text{ to } \text{left}(o_2) \text{ in } G^*.
  \item There is a path from \text{right}(o_2) \text{ to } \text{left}(o_1) \text{ in } G^*.
\end{itemize}

\textbf{Proof:} \quad \text{Exercise!}
Tessellation / Tiling

- Tiles: axis-parallel rectangles

- can be unbounded, or degenerate (line segment/point)

- $\theta_1$, $\theta_2$ horizontally/vertically adjacent $\iff$ common vertical/horizontal boundary

- we write $\theta = [x_1(\theta), x_2(\theta)] \times [y_1(\theta), y_2(\theta)]$
Tessellation / Tiling

**Def.** A *tessellation* $\theta$ of a planar *st*-graph $G$ places each object $o \in V \cup E \cup F$ onto a *tile* $\theta(o)$, so that:

\[ o_1 \neq o_2 \implies \text{int}(\theta(o_1)) \cap \text{int}(\theta(o_2)) = \emptyset \]

\[ \bigcup_{o \in V \cup E \cup F} \theta(o) \text{ is a rectangle.} \]

\[ \theta(o_1) \text{ and } \theta(o_2) \text{ horizontally adjacent } \iff \]

\[ o_1 = \text{left}(o_2) \text{ or } o_1 = \text{right}(o_2) \text{ or } o_2 = \text{left}(o_1) \text{ or } o_2 = \text{right}(o_1) \]

(neither $o_1$ nor $o_2$ has distinct neighbours!)

\[ \theta(o_1) \text{ and } \theta(o_2) \text{ vertically adjacent } \iff \]

\[ o_1 = \text{orig}(o_2) \text{ or } o_1 = \text{dest}(o_2) \text{ or } o_2 = \text{orig}(o_1) \text{ or } o_2 = \text{dest}(o_1) \]

(neither $o_1$ nor $o_2$ has distinct neighbours!)
Tessellation Algorithm (for a Planar \( st \)-Graph \( G \))

\( \gg \) Compute the dual \( G^* \).
\( \gg \) Compute topological numberings \( X \) of \( G^* \) and \( Y \) of \( G \).
\( \gg \) For each object \( o \in V \cup E \cup F \), set \( \theta(o) = [X(\text{left}(o)), X(\text{right}(o))] \times [Y(\text{orig}(o)), Y(\text{dest}(o))] \).
Tessellation Algorithm (for a Planar st-Graph $G$)

\[
\begin{align*}
\text{Compute the dual } & G^*. \\
\text{Compute topological numberings } & X \text{ of } G^* \text{ and } Y \text{ of } G. \\
\text{For each object } o \in V \cup E \cup F, \text{ set} \\
& \theta(o) = [X(\text{left}(o)), X(\text{right}(o))] \times [Y(\text{orig}(o)), Y(\text{dest}(o))].
\end{align*}
\]
Tessellation Algorithm (for a Planar $st$-Graph $G$)

- Compute the dual $G^*$.
- Compute topological numberings $X$ of $G^*$ and $Y$ of $G$.
- For each object $o \in V \cup E \cup F$, set
  \[
  \theta(o) = [X(\text{left}(o)), X(\text{right}(o))] \times [Y(\text{orig}(o)), Y(\text{dest}(o))].
  \]

**Correctness:**

- Lemma 4 guarantees disjointness.
- Neighbourhood conditions follow from the coordinate mapping.

**Runtime:** $O(n)$
Size Conditions

Minimum height/width $h, w : E \rightarrow \mathbb{R}_{\geq 0}$ for each edge tile.

- Compute optimal *weighted* topological numberings $Y$ of $G = (V, E; h)$ and $X$ of $G^* = (F, E^*; w)$.

- Vertex/face tiles: modify $G$ to $G'$

  - Now each object of $G$ corresponds to an edge in $G'$.

**Thm:** A minimum area tessellation of a planar $st$-graph $G$ with minimum height/width $h, w : V \cup E \cup F \rightarrow \mathbb{R}_{\geq 0}$ can be computed in $O(n)$ time.
Visibility Representations

**Def.** A visibility representation \( \Gamma \) of a planar \( st \)-graph \( G \) has

- vertex \( v \) as a horizontal segment \( \Gamma(v) \)
- and edge \( (u, v) \) as a vertical segment \( \Gamma(u, v) \)

such that

- vertex segments are pairwise disjoint,
- edge segments are pairwise disjoint, and
- the edge segment \( \Gamma(u, v) \) starts from the top of the vertex segment \( \Gamma(u) \), ends on the bottom of vertex segment \( \Gamma(v) \), and does not intersect other vertex segments.
Computing a Visibility Representation

Use the tessellation: vertices are degenerate (i.e., line segments); faces are not degenerate
Algorithm Visibility(planar st-graph G)

- Compute the dual $G^\star$.
- Compute optimal weighted topological numberings $Y$ of $G$ and $X$ of $G^\star$ with unit weights.
- For each vertex $v \in V$, set $\Gamma(v) = [X(left(v)), X(right(v)) - 1] \times \{Y(v)\}$.
- For each edge $e \in E$, set $\Gamma(e) = \{X(left(e))\} \times [Y(orig(e)), Y(dest(e))]$. 
Algorithm Visibility(planar st-graph $G$)

- Compute the dual $G^*$. 
- Compute optimal weighted topological numberings $Y$ of $G$ and $X$ of $G^*$ with unit weights. 
- For each vertex $v \in V$, set $\Gamma(v) = [X(\text{left}(v)), X(\text{right}(v)) - 1] \times \{Y(v)\}$. 
- For each edge $e \in E$, set $\Gamma(e) = \{X(\text{left}(e))\} \times [Y(\text{orig}(e)), Y(\text{dest}(e))]$. 

**Thm:** In $O(n)$ time, the Visibility algorithm generates a visibility representation with integer coordinates and area $O(n^2)$. 