Algorithms for Graph Visualization

Summer Semester 2017

Lecture #4

Upward Planar Drawings

(based on slides from Martin Nöllenburg and Robert Görke, KIT)
The Problem

Definition.

A directed Graph \( D = (V,A) \) is upward planar, when it has a drawing such that:

- all edges are upward y-monotone curves, and
- no two edges cross.

Obvious requirements?

- Planar & acyclic.

not sufficient!
Problem: Upward Planarity Testing

Given a directed acyclic graph $D = (V, A)$. Determine if $D$ is upward planar. If so, construct a corresponding drawing.

NP-hard! [Garg & Tamassia '95]

Problem': Embedded Upward Planarity Testing

Given an acyclic graph $D = (V, A)$ with an embedding $F, f_0$. Determine if $D$ is upward planar with respect to $F, f_0$. If so, construct a corresponding drawing.

Can be tested efficiently! [this lecture]
The Big Picture: a characterization

**Theorem** [Kelly ’87, Di Battista & Tamassia ’88]

For a directed graph $D = (V, A)$, the following are equivalent.

1. $D$ is upward planar.
2. $D$ has a *straight-line* upward planar drawing.
3. $D$ is a spanning subgraph of a planar *st-graph*.

Additionally:
- embedded so that $s$ and $t$ are on the outer-face $f_0$.
- acyclic directed graph with a single source $s$ and single sink $t$.
- without crossings
The Big Picture: a characterization

**Theorem** [Kelly '87, Di Battista & Tamassia '88]

For a directed graph \( D = (V, A) \), the following are equivalent.

1. \( D \) is upward planar.
2. \( D \) has a *straight-line* upward planar drawing.
3. \( D \) is a spanning subgraph of a planar st-graph.

*Proof in textbook* [DETT, Sec. 6.1]

\( D \) can be drawn upward planar, see textbook [DETT, Sec. 6.1]
Bimodality

bimodal vertex  \hspace{1cm} not bimodal

**Definition**
An embedded directed graph is \textit{bimodal} \iff all vertices are bimodal.

**Lemma**
An embedded directed graph is upward planar only if it is bimodal.
Angle Sizes of Sources and Sinks

For a face $f$ of a straight-line drawing, consider angles of
– local sinks (vertices with 2 incoming edges on $\partial f$)
– local sources (vertices with 2 outgoing edges on $\partial f$)

$\Rightarrow L(f) := \text{number of large angles}$ (Intuition: in drawing $> \pi$)
$\Rightarrow S(f) := \text{number of small angles}$
$\Rightarrow A(f) := \text{number of local sources} = \text{number of local sinks}$

Thus:
$L(f) + S(f) = 2A(f)$

By induction:
$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases} \Rightarrow L(f) = \begin{cases} A(f) - 1, & f \neq f_0 \\ A(f) + 1, & f = f_0 \end{cases}$
Proof: \( L(f) - S(f) = -2 \) for \( f \neq f_0 \)

\[ \Rightarrow L(f) = 0 \quad \Rightarrow S(f) = 2 \]

\[ \Rightarrow L(f) \geq 1 \]

Separate \( f \) by.

5. \( v \) neither source nor sink:

\[
L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1)
= -2
\]

induction hypothesis
### Observations

Consider the angle between two incoming/outgoing edges.

### Lemma

Let $D$ be a directed graph. In every upward planar drawing of $D$:

1. For each vertex $v \in V$: $L(v) = \begin{cases} 0 & \text{v inner vertex,} \\ 1 & \text{v source/sink.} \end{cases}$

2. For each face $f \in F$: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$

$\Phi: S \cup T \rightarrow F$ called **consistent**

\[ |\Phi^{-1}(f)| = \begin{cases} A(f) - 1 & f \neq f_0 \\ A(f) + 1 & f = f_0 \end{cases} \]

\(\Phi\): \(S \cup T \rightarrow F\)

$v \mapsto$ incid. face
called consistent

\(global\) sources and sinks
Example: Face Assignment

Assignment \( \phi : S \cup T \rightarrow F \)

- \( A(f_0) = 3 \) 
  \( L(f_0) = 4 \)

- \( A(f_1) = 3 \) 
  \( L(f_1) = 2 \)

- \( A(f_2) = 1 \) 
  \( L(f_2) = 0 \)

- \( A(f_3) = 1 \) 
  \( L(f_3) = 0 \)

- \( A(f_4) = 2 \) 
  \( L(f_4) = 1 \)

- \( A(f_5) = 2 \) 
  \( L(f_5) = 1 \)

- \( A(f_6) = 1 \) 
  \( L(f_6) = 0 \)

- \( A(f_7) = 2 \) 
  \( L(f_7) = 1 \)

- \( A(f_8) = 1 \) 
  \( L(f_8) = 0 \)

- \( A(f_9) = 1 \) 
  \( L(f_9) = 0 \)
Main Result

**Theorem**

If $D = (V, A)$ is a dir. acyclic graph with embedding $\mathcal{F}, f_0$. Then:

$D$ upward planar (resp. $\mathcal{F}, f_0$) $\iff$ bimodal and $\exists$ consistent $\Phi$.

$\Rightarrow$: as constructed before

$\Leftarrow$: ideas

- construct st-Graph $\supseteq D$
  - apply equivalence from the beginning of the lecture

First: $D, \mathcal{F}, f_0 \rightarrow ? \Phi$ consistent assignment
Flow Network to Construct $\Phi$

**Definition Flow Network**

$$N_{\mathcal{F}, f_0}(D) = ((W, A_N); l; u; d)$$

- $W = \{ v \in V \mid v \text{ is source or sink} \} \cup \mathcal{F}$
- $A_N = \{ (v, f) \mid v \text{ incident to } f \}$
- $l(a) = 0 \quad \forall a \in A_N$
- $u(a) = 1 \quad \forall a \in A_N$
- $d(q) = \begin{cases} 
1 & \forall q \in W \cap V \\
-(A(q) - 1) & \forall q \in \mathcal{F} \setminus \{f_0\} \\
-(A(q) + 1) & q = f_0
\end{cases}$

idea: flow $(v, f) = 1$ iff $v$ is a global source/sink whose large angle is assigned to $f$
Example Network

- normal vertex
- source / sink
- face vertex
Algorithm: $\Phi, \mathcal{F}, f_0 \rightarrow \text{st-Graph} \supseteq D$

Let $f$ be a face. Consider the clockwise angle sequence $\sigma_f$ of $L/S$ on local sources and sinks of $f$.

- Goal: Add edges to break large angles (sources and sinks).

- $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle L, S, S \rangle$ at vertices $x, y, z$

- $x$ source $\Rightarrow$ insert edge $(z, x)$

- $x$ sink $\Rightarrow$ insert edge $(x, z)$

- Refine the outerface $f_0$

Refine all $f \in \mathcal{F} \Rightarrow D$ is contained in a planar st-Graph
Example Refinement
Example Refinement
Summary

Given: embedded, directed, acyclic graph $G = (V, E)$.

Test for bimodality

Test for a consistent assignment $\Phi$ (via flow network).

If both bimodal and $\Phi$ exists, draw $G$ as upward planar.

- refine $G$ to planar $st$-graph $G'$
- Draw $G'$ via $st$-graph methods
- Delete the edges added by refinement.

gives up. planar drawing, see textbook [DETT, Sec. 6.1] – but the area usage can be exponential!
Finding the angles via the flow network

\[ W := V \cup \mathcal{F} \]
\[ A := \{(v, f) \in V \times \mathcal{F} : v \text{ incident (\sim) to } f\} \]
\[ \ell(a) = 0 \quad \forall a \in A \]
\[ u(a) = 2\pi \quad \forall a \in A \]
\[ d(v) = 2\pi \quad \forall v \in V \]
\[ d(f) = \begin{cases} 
-(\text{deg}(f) - 2)\pi & \text{if } f \neq f_0, \\
-(\text{deg}(f) + 2)\pi & \text{otherwise} 
\end{cases} \]

Flow provides an assignment \( x(\cdot, \cdot) \) of angles where:

1. vertices : \( \forall v \in V : \sum_{f \sim v} x(v, f) = 2\pi \)
2. faces : \( \forall f \in \mathcal{F} : \sum_{v \sim f} x(v, f) = (\text{deg}(f) \pm 2)\pi \)

1. and 2. mean: assignment \textit{locally consistent}.

\textbf{Obs.} using edge costs we can maximize \textit{angular resolution}. (via Linear Programming)
Locally Consistent $\not\Rightarrow$ Globally Consistent

- Not isoceles!
Characterizing Inner Triangulations

**Theorem** [Di Battista & Vismara '93]

Given planar inner triangulation\(^*\) with embedding \(\mathcal{F}, f_0\) and angle assignment \(x\), then:

There is a straight-line drawing with \(\mathbb{R}^2 \setminus f_0\) convex if and only if:

\[
\begin{align*}
1. & \sum \text{ vertex angles} = 2\pi \\
2. & \sum \text{ face angles} = \pi \\
3. & \text{for every } v \not\sim f_0, \text{ via radius } R_v: \prod_{i=1}^{\deg v} \frac{\sin \alpha_i}{\sin \beta_i} = 1 \\
4. & \text{for every } v \sim f_0, \sum_{v \sim f \neq f_0} x(v, f) \leq \pi
\end{align*}
\]

*Problem*: it’s not a linear condition :-(

\(^*\) Every face \(f \neq f_0\) is a triangle \((C_3)\).