Algorithms for Graph Visualization

Summer Semester 2017
Lecture #4

Upward Planar Drawings

(based on slides from Martin Nöllenburg and Robert Görke, KIT)
The Problem

**Definition.**

A directed Graph $D = (V, A)$ is *upward planar*, when it has a drawing such that:

- all edges are upward y-monotone curves, and
- no two edges cross.
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- Planar & acyclic.
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not sufficient!
Upward Planarity

Problem: Upward Planarity Testing

Given a directed acyclic graph $D = (V, A)$. Determine if $D$ is upward planar. If so, construct a corresponding drawing.
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Problem’: Embedded Upward Planarity Testing

Given an acyclic graph $D = (V, A)$ with an embedding $F, f_0$. Determine if $D$ is upward planar with respect to $F, f_0$. If so, construct a corresponding drawing.
Upward Planarity

Problem: Upward Planarity Testing

Given a directed acyclic graph $D = (V, A)$. Determine if $D$ is upward planar. If so, construct a corresponding drawing.

NP-hard! [Garg & Tamassia ’95]

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Upward Planarity

**Problem:** Upward Planarity Testing

Given a directed acyclic graph $D = (V, A)$.
Determine if $D$ is upward planar.
If so, construct a corresponding drawing.

⇒ NP-hard! [Garg & Tamassia ’95]

**Problem’: Embedded Upward Planarity Testing**

Given an acyclic graph $D = (V, A)$ with an embedding $\mathcal{F}, f_0$.
Determine if $D$ is upward planar with respect to $\mathcal{F}, f_0$.
If so, construct a corresponding drawing.

⇒ Can be tested efficiently! [this lecture]
Theorem \cite{Kelly '87, Di Battista & Tamassia '88}

For a directed graph $D = (V, A)$, the following are equivalent.

1. $D$ is upward planar.
2. $D$ has a \textit{straight-line} upward planar drawing.
3. $D$ is a spanning subgraph of a planar st-graph.
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The Big Picture: a characterization

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Without crossings
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---

without crossings

acyclic directed graph with a single source $s$ and single sink $t$. 
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Additionally:
- without crossings
- acyclic directed graph with a single source $s$ and single sink $t$.

embedded so that $s$ and $t$ are on the outer-face $f_0$. 

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Steven Chaplick · Lehrstuhl für Informatik I · Universität Würzburg
The Big Picture: a characterization

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*Proof in textbook* [DETT, Sec. 6.1]
Theorem \cite{Kelly87,DiBattistaTamassia88}

For a directed graph $D = (V, A)$, the following are equivalent.

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\textit{Proof in textbook} \cite{DETT,Sec.6.1}

\textit{can be drawn upward planar, see textbook} \cite{DETT,Sec.6.1}
Bimodality

bimodal vertex

![Diagram of a bimodal vertex](image-url)
Bimodality

bimodal vertex

not bimodal
**Bimodality**

**bimodal vertex**

**not bimodal**

**Definition**

An embedded directed graph is *bimodal* if and only if all vertices are bimodal.

all vertices are bimodal.
Bimodality

**bimodal vertex**    **not bimodal**

---

**Definition**
An embedded directed graph is *bimodal* if all vertices are bimodal.

---

**Lemma**
An embedded directed graph is upward planar only if it is bimodal.
Angle Sizes of Sources and Sinks

For a face $f$ of a straight-line drawing, consider angles of
– local sinks (vertices with 2 incoming edges on $\partial f$)
– local sources (vertices with 2 outgoing edges on $\partial f$)
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$\gg L(f) := \text{number of large angles} \quad (\text{Intuition: in drawing } > \pi)$
$\gg S(f) := \text{number of small angles}$
$\gg A(f) := \text{number of local sources} \ (= \text{number of local sinks})$
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Thus:
$L(f) + S(f) =$
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Thus:
$L(f) + S(f) = 2A(f)$
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Thus:

$$L(f) + S(f) = 2A(f)$$

By induction:

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$
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Thus:
$L(f) + S(f) = 2A(f)$

By induction:
$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases} \Rightarrow L(f) = \begin{cases} A(f) - 1, & f \neq f_0 \\ A(f) + 1, & f = f_0 \end{cases}$
Proof: $L(f) - S(f) = -2$ for $f \neq f_0$

$\Rightarrow L(f) = 0$
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$\Rightarrow L(f) = 0 \quad \Rightarrow S(f) = 2$
Proof: $L(f) - S(f) = -2$ for $f \neq f_0$

$\implies L(f) = 0 \quad \implies S(f) = 2$

$\implies L(f) \geq 1$
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Separate $f$ by.
Proof: \( L(f) - S(f) = -2 \) for \( f \neq f_0 \)

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Separate \( f \) by.

1. \( v \) sink with a small angle:
Proof: $L(f) - S(f) = -2$ for $f \neq f_0$

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\[ L(f) \geq 1 \]

Separate $f$ by.

1. $v$ sink with a small angle:

\[
L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1) = -2
\]
Proof: $L(f) - S(f) = -2$ for $f \neq f_0$

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Proof: \( L(f) - S(f) = -2 \) for \( f \neq f_0 \)

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Separate \( f \) by.

2. \( v \) sink with a big angle:

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L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1) = -2
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induction hypothesis
Proof: \( L(f) - S(f) = -2 \) for \( f \neq f_0 \)

\[ L(f) = 0 \quad \Rightarrow \quad S(f) = 2 \]

\[ L(f) \geq 1 \]

Separate \( f \) by.

3. \( v \) source with big angle:

\[ L(f) - S(f) = L(f_1) + L(f_2) + 2 - (S(f_1) + S(f_2)) = -2 \]

induction hypothesis
Proof: $L(f) - S(f) = -2$ for $f \neq f_0$

$\Rightarrow L(f) = 0 \quad \Rightarrow S(f) = 2$

$\Rightarrow L(f) \geq 1$

Separate $f$ by.

4. $v$ source with small angle:
Proof: $L(f) - S(f) = -2$ for $f \neq f_0$

\[ L(f) = 0 \quad \Rightarrow \quad S(f) = 2 \]

\[ L(f) \geq 1 \]

Separate $f$ by.

5. $v$ neither source nor sink:

\[
L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1)
\]
\[ = -2 \]

induction hypothesis
Observations

Consider the angle between two incoming/outgoing edges.
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Consider the angle between two incoming/outgoing edges.

**Lemma**

Let $D$ be a directed graph. In every upward planar drawing of $D$:

(1) for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source/sink.} \end{cases}$

(2) for each face $f \in F$: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$
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2. for each face $f \in \mathcal{F}$: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$

$\Phi: S \cup T \rightarrow \mathcal{F}$

$v \mapsto$ incid. face

$|\Phi^{-1}(f)| = \text{global sources and sinks}$
Observations

Consider the angle between two incoming/outgoing edges.

**Lemma**

Let $D$ be a directed graph. In every upward planar drawing of $D$:

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$\Phi : S \cup T \rightarrow F$

$v \mapsto \text{incid. face}$ called *consistent* global sources and sinks

$|\Phi^{-1}(f)| = L(f)$
Observations

Consider the angle between two incoming/outgoing edges.

Lemma

Let \( D \) be a directed graph.
In every upward planar drawing of \( D \):

1. for each vertex \( v \in V \): \( L(v) = \begin{cases} 0 & \text{v inner vertex,} \\ 1 & \text{v source/sink.} \end{cases} \)

2. for each face \( f \in F \): \( L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases} \)

\( \Phi : S \cup T \rightarrow F \) called \textit{consistent} global sources and sinks

\( \Phi^{-1}(f) = \begin{cases} A(f) - 1 & f \neq f_0 \\ A(f) + 1 & f = f_0 \end{cases} \)
Example: Face Assignment
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- Global sources and sinks
Example: Face Assignment

A(f_0) = 3
A(f_1) = 3
A(f_2) = 1
A(f_3) = 1
A(f_4) = 2
A(f_5) = 2
A(f_6) = 1
A(f_7) = 2
A(f_8) = 1
A(f_9) = 1
Example: Face Assignment

- \( A(f_1) = 3 \)
  \( L(f_1) = 2 \)

- \( A(f_2) = 1 \)
  \( L(f_2) = 0 \)

- \( A(f_3) = 1 \)
  \( L(f_3) = 0 \)

- \( A(f_4) = 2 \)
  \( L(f_4) = 1 \)

- \( A(f_5) = 2 \)
  \( L(f_5) = 1 \)

- \( A(f_6) = 1 \)
  \( L(f_6) = 0 \)

- \( A(f_7) = 2 \)
  \( L(f_7) = 1 \)

- \( A(f_8) = 1 \)
  \( L(f_8) = 0 \)

- \( A(f_9) = 1 \)
  \( L(f_9) = 0 \)

Global sources and sinks
Example: Face Assignment

Assignment $\phi : S \cup T \to \mathcal{F}$

- $A(f_1) = 3$
- $L(f_1) = 2$
- $A(f_2) = 1$
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- $A(f_3) = 1$
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- $A(f_9) = 1$
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Main Result

Theorem

If $D = (V, A)$ is a dir. acyclic graph with embedding $\mathcal{F}, f_0$. Then:
$D$ upward planar (resp. $\mathcal{F}, f_0$) $\iff$ bimodal and $\exists$ consistent $\Phi$. 
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If $D = (V, A)$ is a dir. acyclic graph with embedding $\mathcal{F}, f_0$. Then:

\[ D \text{ upward planar (resp. } \mathcal{F}, f_0) \iff \text{bimodal and } \exists \text{ consistent } \Phi. \]

$\Rightarrow$: as constructed before

$\Leftarrow$: ideas
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$\Leftarrow$: ideas
  - construct st-Graph $\supseteq D$
Main Result

**Theorem**

If $D = (V, A)$ is a dir. acyclic graph with embedding $F, f_0$. Then:

$D$ upward planar (resp. $F, f_0$) ⇔ bimodal and $\exists$ consistent $\Phi$.

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– construct st-Graph $\supseteq D$

– apply equivalence from the beginning of the lecture
Main Result

**Theorem**

If $D = (V, A)$ is a dir. acyclic graph with embedding $F, f_0$. Then:

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$\Leftarrow$: ideas

- construct st-Graph $\supseteq D$

- apply equivalence from the beginning of the lecture

First: $D, F, f_0 \rightarrow \Phi$ consistent assignment
Flow Network to Construct $\Phi$

**Definition Flow Network**

$N_{\mathcal{F}, f_0}(D) = ((W, A_N); l; u; d)$

- $W = \{v \in V \mid v \text{ is source or sink}\} \cup \mathcal{F}$
- $A_N = \{(v, f) \mid v \text{ incident to } f\}$
- $l(a) = 0 \quad \forall a \in A_N$
- $u(a) = 1 \quad \forall a \in A_N$
- $d(q) =
  \begin{cases} 
  1 & \forall q \in W \cap V \\
  -(A(q) - 1) & \forall q \in \mathcal{F} \setminus \{f_0\} \\
  -(A(q) + 1) & q = f_0
  \end{cases}$

**Idea:** flow $(v, f) = 1$ iff $v$ is a global source/sink whose large angle is assigned to $f$
Example Network

- normal vertex
- source / sink
Example Network

- normal vertex
- source / sink
- face vertex
Example Network

- normal vertex
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- face vertex
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Algorithm: \( \Phi, \mathcal{F}, f_0 \rightarrow \text{st-Graph} \supseteq D \)
Algorithm: $\Phi, \mathcal{F}, f_0 \rightarrow \text{st-Graph} \supseteq D$

Let $f$ be a face. Consider the clockwise angle sequence $\sigma_f$ of $\text{L}/\text{S}$ on local sources and sinks of $f$. 
Algorithm: $\Phi, \mathcal{F}, f_0 \rightarrow \text{st-Graph} \subseteq D$

Let $f$ be a face. Consider the clockwise angle sequence $\sigma_f$ of L/S on local sources and sinks of $f$

Goal: Add edges to break large angles (sources and sinks).
Algorithm: $\Phi, \mathcal{F}, f_0 \rightarrow \text{st-Graph} \subseteq D$

Let $f$ be a face. Consider the clockwise angle sequence $\sigma_f$ of $L/S$ on local sources and sinks of $f$.

Goal: Add edges to break large angles (sources and sinks).

$f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle L, S, S \rangle$ at vertices $x, y, z$. 

![Graph diagram with labels L and S at vertices, showing angles σ.](attachment:grafik.png)
Algorithm: $\Phi, \mathcal{F}, f_0 \rightarrow \text{st-Graph} \subseteq \mathcal{D}$

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$x$ source $\Rightarrow$ insert edge $(z, x)$
Algorithm: $\Phi, \mathcal{F}, f_0 \rightarrow \text{st-Graph} \supseteq D$

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$\triangleright \triangleright$ Goal: Add edges to break large angles (sources and sinks).

$\triangleright \triangleright f \neq f_0 \text{ with } |\sigma_f| \geq 2 \text{ containing } \langle L, S, S \rangle \text{ at vertices } x, y, z$

$\triangleright \triangleright x \text{ source } \Rightarrow \text{ insert edge } (z, x)$
Algorithm: \( \Phi, \mathcal{F}, f_0 \rightarrow \text{st-Graph} \subseteq D \)

Let \( f \) be a face. Consider the clockwise angle sequence \( \sigma_f \) of \( L/S \) on local sources and sinks of \( f \).

\[ \Rightarrow \] **Goal:** Add edges to break large angles (sources and sinks).

\[ \Rightarrow \] \( f \neq f_0 \) with \( |\sigma_f| \geq 2 \) containing \( \langle L, S, S \rangle \) at vertices \( x, y, z \)

\[ \Rightarrow \] \( x \) source \( \Rightarrow \) insert edge \( (z, x) \)
Algorithm: $\Phi, \mathcal{F}, f_0 \rightarrow \text{st-Graph} \supseteq D$

Let $f$ be a face. Consider the clockwise angle sequence $\sigma_f$ of L/S on local sources and sinks of $f$.

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$\implies$ Refine the outerface $f_0$
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$\Rightarrow$ Refine the outerface $f_0$

Refine all $f \in \mathcal{F} \Rightarrow D$ is contained in a planar st-Graph
Example Refinement
Example Refinement
Example Refinement
Example Refinement
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Summary

Given: embedded, directed, acyclic graph $G = (V, E)$. 
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\( \Rightarrow \) Test for bimodality

\( \Rightarrow \) Test for a consistent assignment \( \Phi \) (via flow network).

\( \Rightarrow \) If both bimodal and \( \Phi \) exists, draw \( G \) as upward planar.
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  - Draw \( G' \) via \( st \)-graph methods
  - Delete the edges added by refinement.
Summary

Given: embedded, directed, acyclic graph $G = (V, E)$.

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See textbook [DETT, Sec. 6.1] for planar drawing.
Summary

Given: embedded, directed, acyclic graph $G = (V, E)$.

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If both bimodal and $\Phi$ exists, draw $G$ as upward planar.

- refine $G$ to planar $st$-graph $G'$
- Draw $G'$ via $st$-graph methods
- Delete the edges added by refinement.

See textbook [DETT, Sec. 6.1] — but the area usage can be exponential!
Finding the angles via the flow network

\[ W := V \cup \mathcal{F} \]
Finding the angles via the flow network

\[ W := V \cup F \]
\[ A := \{(v, f) \in V \times F : v \text{ incident } (\sim) \text{ to } f\} \]
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Flow provides an assignment \(x(\cdot, \cdot)\) of angles where:

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1. and 2. mean: assignment \emph{locally consistent}. 
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1. and 2. mean: assignment \textit{locally consistent}.

\textbf{Obs.} using edge costs we can maximize \textit{angular resolution}. 
Locally Consistent $\not\Rightarrow$ Globally Consistent
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Characterizing Inner Triangulations

Theorem [Di Battista & Vismara '93]

Given planar inner triangulation* with embedding $\mathcal{F}, f_0$ and angle assignment $x$, then:

1. $\sum$ vertex angles $= 2\pi$
2. $\sum$ face angles $= \pi$
3. for every $v \simneq f_0$, via radius $R_v$: $\prod \deg v_i \sin \alpha_i \sin \beta_i = 1$
4. for every $v \sim f_0$, $\sum_{v \sim \neq f_0} x(v,f) \leq \pi$

*) Every face $f \neq f_0$ is a triangle ($C_3$).
Characterizing Inner Triangulations

**Theorem** [Di Battista & Vismara ’93]

Given planar inner triangulation* with embedding $\mathcal{F}, f_0$ and angle assignment $x$, then:

There is a straight-line drawing with $\mathbb{R}^2 \setminus f_0$ convex

\[
\begin{cases}
1. \sum \text{ vertex angles} = 2\pi \\
2. \sum \text{ face angles} = \pi \\
\end{cases}
\]

$\Leftrightarrow$

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Characterizing Inner Triangulations

Theorem [Di Battista & Vismara ’93]

Given planar inner triangulation* with embedding $\mathcal{F}, f_0$ and angle assignment $x$, then:

There is a straight-line drawing with $\mathbb{R}^2 \setminus f_0$ convex if and only if:

\[
\begin{align*}
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    3. \quad \text{for every } v \sim f_0, \text{ via radius } R_v: \prod_{i=1}^{\deg v} \frac{\sin \alpha_i}{\sin \beta_i} = 1 \\
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*Problem: it’s not a linear condition :-(

*) Every face \( f \neq f_0 \) is a triangle \( (C_3) \).