Algorithms for Graph Visualization

Summer Semester 2017
Lecture #1

Divide-and-Conquer Algorithms:
Trees and Series-Parallel Graphs

(based on slides from Martin Nöllenburg and Robert Görke, KIT)
Uses of Divide & Conquer

Well suited for inductively/recursively defined Graph Classes
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Rooted Binary Trees:

1. draw the left subtree
2. draw the right subtree
3. combine together + draw root
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Terminology

- depth(v): distance from the root
- traversal
  - preorder
  - inorder
  - postorder
Overview

- balanced drawings of binary trees $O(nh)$
- radial drawings of trees $O(nh)$
- compact drawings of trees $O(n \log n)$
- upward drawings of series parallel graphs $\text{exp.}$
Overview

- balanced drawings of binary trees
  \(O(nh)\)
- radial drawings of trees
  \(O(nh)\)
- compact drawings of trees
  \(O(n \log n)\)
- upward drawings of series parallel graphs
  \(\exp\)
Algorithm of Reingold and Tilford ('81)

Trivial: If $T = \{v\}$, draw $v$ (e.g., as a small disk).
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place the root $r$ one above and centrally btw. children.
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Grid Drawing?

or 3!
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Implementation in 2 Phases:

1. postorder (bottom-up):
   - contours and x-offsets
   - gather the predecessors

2. preorder (top-down):
   - calculate absolute coordinates
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Contour: linked list of vertices (-coordinates)
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1. compute $T_\ell(v)$ und $T_r(v)$
2. trace the right contour of $T_\ell(v)$ and left of $T_r(v)$
3. Find $d_v = \text{min. horiz. distance between } v_\ell \text{ und } v_r$
4. $x$-offset($v_\ell$) = $−\lceil d_v/2 \rceil$, $x$-offset($v_r$) = $\lceil d_v/2 \rceil$
5. Build left contour of $T_v$ from:
   $\gg v$,
   $\gg$ left contour of $T_\ell(v)$,
   $\gg$ left contour of any low hanging part of $T_r(v)$
6. Symmetrically for right contour.
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Runtime?
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Phase 2:

Set \( y \)-coordinate \( y(v) = -\text{depth}(v) \) for each vertex \( v \).
Algorithm of Reingold und Tilford ('81)

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- Set \( y(v) = -\text{depth}(v) \) for each vertex \( v \).
- Set \( x(w) := 0 \) for the root \( w \), then in preorder for \( v \in V \):
  - \( x(v_\ell) := x(v) + \text{x-offset}(v_\ell) \) and
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Runtime? $O(n)$
Summary for Balanced Drawings of Binary Trees

**Theorem** [Reingold & Tilford ’81]

For a binary tree with $n$ vertices, in $O(n)$ time we can produce a drawing $\Gamma$ such that:
Summary for Balanced Drawings of Binary Trees

Theorem [Reingold & Tilford ’81]

For a binary tree with \( n \) vertices, in \( O(n) \) time we can produce a drawing \( \Gamma \) such that:

- \( \Gamma \) is layered, i.e., \( y \equiv -\text{depth} \),
- \( \Gamma \) is planar, straightline, and strictly downward,
- \( \Gamma \) matches the embedding (i.e., right children on the right),
- all vertices: horiz. \& vert. distances \( \geq 1 \), and on the grid,
- the area is \( O(n^2) \),
- parent always centered above children.
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Example?
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Easily generalizes to arbitrary trees!
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Min. width (but without the grid):
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\end{itemize}

Min. width (but without the grid): by LP!

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# Summary for Balanced Drawings of Binary Trees

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Min. width and on the grid: NP-hard! [Supowit & Reingold '83]
Example of width variation

Output of the Algorithm:
Example of width variation

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Output of the Algorithm:

Optimal Drawing:
Example of width variation

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Optimal Drawing:
2. Radial Drawings of Trees

- balanced drawings of binary trees $O(nh)$
- radial drawings of trees $O(nh)$
- compact drawings of trees $O(n \log n)$
- upward drawings of series parallel graphs $\exp.$
Example: Radial Tree Layouts
An Algorithm for Radial Layout?
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Restricting to Smaller Sectors
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Restricting to Smaller Sectors

\[ \cos \tau = \frac{\rho_i}{\rho_{i+1}} \]

\[ \Rightarrow \begin{cases} 
\alpha_{\text{min}} &= \alpha_v - \arccos \left( \frac{\rho_i}{\rho_{i+1}} \right) \\
\alpha_{\text{max}} &= \alpha_v + \arccos \left( \frac{\rho_i}{\rho_{i+1}} \right)
\end{cases} \]
Pseudocode for radial tree layout

RadialTreeLayout(tree \( T \), root \( r \in T \), radii \( \rho_1 < \cdots < \rho_k \))

\[\begin{align*}
\text{begin} \\
\quad \text{postorder}(r) \\
\quad \text{preorder}(r, 0, 0, 2\pi) \\
\quad \text{return } (d_v, \alpha_v)_{v \in V(T)} \\
\quad \{\text{vertex pos./ polar coord.}\}
\end{align*}\]

\[\begin{align*}
\text{postorder(vertex } v) \\
\quad \text{calculate the size of the subtree recursively}
\end{align*}\]
Pseudocode for radial tree layout

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end

postorder(vertex $v$)

$n_v \leftarrow 1$

foreach child $w$ von $v$ do
  postorder($w$)
  $n_v \leftarrow n_v + n_w$
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\textit{size of the subtree $T(v)$}
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\end{align*} \]

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\end{align*} \]

size of the subtree $T(v)$

preorder(vertex $v$, $t$, $\alpha_{\text{min}}$, $\alpha_{\text{max}}$)

\[ \begin{align*}
\quad & d_v \leftarrow \rho_t \\
\quad & \alpha_v \leftarrow (\alpha_{\text{min}} + \alpha_{\text{max}})/2 \\
\quad & \text{if } t > 0 \text{ then} \\
\quad & \quad \alpha_{\text{min}} \leftarrow \max\{\alpha_{\text{min}}, \alpha_v - \arccos \frac{\rho_t}{\rho_{t+1}}\} \\
\quad & \quad \alpha_{\text{max}} \leftarrow \min\{\alpha_{\text{max}}, \alpha_v + \arccos \frac{\rho_t}{\rho_{t+1}}\} \\
\quad & \quad \text{left} \leftarrow \alpha_{\text{min}} \\
\quad & \text{foreach child } w \text{ von } v \text{ do} \\
\quad & \quad \quad \text{right} \leftarrow \text{left} + \frac{n_w}{n_v - 1} \cdot (\alpha_{\text{max}} - \alpha_{\text{min}}) \\
\quad & \quad \quad \text{preorder}(w, t + 1, \text{left}, \text{right}) \\
\quad & \quad \text{left} \leftarrow \text{right}
\end{align*} \]
Pseudocode for radial tree layout

RadialTreeLayout(tree $T$, root $r \in T$, radii $\rho_1 < \cdots < \rho_k$)

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    postorder($r$)
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preorder(vertex $v$, $t$, $\alpha_{\text{min}}$, $\alpha_{\text{max}}$)

$d_v \leftarrow \rho_t$
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if $t > 0$ then
    $\alpha_{\text{min}} \leftarrow \max\{\alpha_{\text{min}}, \alpha_v - \arccos \frac{\rho_t}{\rho_{t+1}}\}$
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left $\leftarrow\alpha_{\text{min}}$
foreach child $w$ von $v$ do
    right $\leftarrow left + \frac{n_w}{n_v-1} \cdot (\alpha_{\text{max}} - \alpha_{\text{min}})$
    preorder($w, t + 1, left, right$)
left $\leftarrow right$
Pseudocode for radial tree layout

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\text{size of the subtree } T(v)

\text{preorder}(vertex $v$, $t$, $\alpha_{\text{min}}$, $\alpha_{\text{max}}$)

\begin{align*}
&d_v \leftarrow \rho_t \\
&\alpha_v \leftarrow (\alpha_{\text{min}} + \alpha_{\text{max}})/2 \\
&\{\text{output}\}
\end{align*}

\text{if } t > 0 \text{ then}

\begin{align*}
&\alpha_{\text{min}} \leftarrow \max\{\alpha_{\text{min}}, \alpha_v - \arccos \frac{\rho_t}{\rho_{t+1}}\} \\
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\text{left} \leftarrow \alpha_{\text{min}}

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\end{align*}
Pseudocode for radial tree layout

RadialTreeLayout(tree $T$, root $r \in T$, radii $\rho_1 < \cdots < \rho_k$)

begin

postorder($r$)
preorder($r$, 0, 0, $2\pi$)
return $(d_v, \alpha_v)_{v \in V(T)}$

{vertex pos./ polar coord.}

postorder(vertex $v$)

$n_v \leftarrow 1$

foreach child $w$ von $v$ do

postorder($w$)

$n_v \leftarrow n_v + n_w$

size of the subtree $T(v)$

preorder(vertex $v$, $t$, $\alpha_{\text{min}}$, $\alpha_{\text{max}}$)

d_v \leftarrow \rho_t

$\alpha_v \leftarrow (\alpha_{\text{min}} + \alpha_{\text{max}})/2$

{output}

if $t > 0$ then

$\alpha_{\text{min}} \leftarrow \max\{\alpha_{\text{min}}, \alpha_v - \arccos \frac{\rho_t}{\rho_{t+1}}\}$

$\alpha_{\text{max}} \leftarrow \min\{\alpha_{\text{max}}, \alpha_v + \arccos \frac{\rho_t}{\rho_{t+1}}\}$

left $\leftarrow \alpha_{\text{min}}$

foreach child $w$ von $v$ do

right $\leftarrow$ left + $\frac{n_w}{n_v-1} \cdot (\alpha_{\text{max}} - \alpha_{\text{min}})$

preorder($w$, $t + 1$, left, right)

left $\leftarrow$ right

Runtime?
Pseudocode for radial tree layout

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  preorder($w, t + 1, \text{left}, \text{right}$)

left $\leftarrow$ right

} {output }

Runtime? $O(n)$. 
Pseudocode for radial tree layout

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preorder($w$, $t + 1$, left, right)

left $\leftarrow right$

end

Runtime? $O(n)$. Correctness?
Pseudocode for radial tree layout

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\{ output \}

if $t > 0$ then

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preorder($w$, $t + 1$, left, right)

left \leftarrow right

Runtime? $O(n)$. Correctness? $\checkmark$
Overview

- balanced drawings of binary trees $O(nh)$
- radial drawings of trees $O(nh)$
- compact drawings of trees $O(n \log n)$
- upward drawings of series parallel graphs $\exp.$
Definition.

An \textit{hv-drawing} of a binary tree is a straight line drawing, so that for each vertex $v$: 
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An *hv-drawing* of a binary tree is a straight line drawing, so that for each vertex $v$:

> each child of $v$ is either directly right or directly below $v$. 
**Definition.**

An *hv-drawing* of a binary tree is a straight line drawing, so that for each vertex $v$:

- each child of $v$ is either directly right or directly below $v$.
- the smallest axis-parallel rectangle enclosing the subtrees of the children of $v$ are disjoint.
hv-Drawings

Definition.

An hv-drawing of a binary tree is a straight line drawing, so that for each vertex \( v \):

\[ \Rightarrow \text{each child of } v \text{ is either directly right or directly below } v. \]

\[ \Rightarrow \text{the smallest axis-parallel rectangle enclosing the subtrees of the children of } v \text{ are disjoint.} \]
**Definition.**

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**hv-drawings**

**horizontal combination**

**vertical combination**
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*horizontal combination* 
*vertical combination*
Algorithm $RightHeavyHVTreeDraw$

\[\rightarrow\text{Recursively construct drawings of the left and right subtrees from the root.}\]
Algorithm *RightHeavyHVTreeDraw*

- Recursively construct drawings of the left and right subtrees from the root.
- Place the larger subtree on the right using the horizontal combination, and the smaller on the left.
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Size of a subtree := number of vertices
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Algorithm \textit{RightHeavyHVTreeDraw}

\begin{itemize}
\item Recursively construct drawings of the left and right subtrees from the root.
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\end{itemize}

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Size of a subtree := number of vertices

**Obs.** The drawing has width \(\leq\)
Algorithm \textit{RightHeavyHVTreeDraw}

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\item Recursively construct drawings of the left and right subtrees from the root.
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\textbf{Obs.} The drawing has width \( \leq n \), height \( \leq \)
Algorithm *RightHeavyHVTreeDraw*

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Size of a subtree := number of vertices

**Obs.** The drawing has width $\leq n$, height $\leq$
Algorithm *RightHeavyHVTREE*Draw

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Size of a subtree := number of vertices

Obs. The drawing has width $\leq n$, height $\leq$
Algorithm RightHeavyHVTreeDraw

- Recursively construct drawings of the left and right subtrees from the root.
- Place the larger subtree on the right using the horizontal combination, and the smaller on the left.

Size of a subtree := number of vertices

Obs. The drawing has width $\leq n$, height $\leq$
Algorithm \textit{RightHeavyHVTeeDraw}

\begin{itemize}
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Algorithm *RightHeavyHVTreeDraw*

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```
  at least \cdot 2
  at least \cdot 2
```
Algorithm *RightHeavyHVTreeDraw*

- Recursively construct drawings of the left and right subtrees from the root.
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Algorithm \textit{RightHeavyHVTreeDraw}

- Recursively construct drawings of the left and right subtrees from the root.
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Size of a subtree := number of vertices

\begin{itemize}
  \item at least \cdot 2
  \item at least \cdot 2
  \item at least \cdot 2
\end{itemize}

\textbf{Obs.} The drawing has width $\leq n$, height $\leq \log_2 n$. 
Overview

- balanced drawings of binary trees $O(nh)$
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- compact drawings of trees $O(n \log n)$
- upward drawings of series parallel graphs $\exp.$
Series Parallel Graphs

simple series parallel graph
Series Parallel Graphs

- simple series parallel graph

- Induction: combining two series parallel graphs $G_1, G_2 \ldots$
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- simple series parallel graph

- Induction: combining two series parallel graphs $G_1, G_2 \ldots$

- \ldots series \ldots

$t_1 = s_2$

\[ s_1 \quad G_1 \quad t_1 \]

\[ s_2 \quad G_2 \quad t_2 \]
Series Parallel Graphs

- simple series parallel graph

- Induction: combining two series parallel graphs $G_1, G_2$ . . .

- ...series...

- ...or parallel.

$t_1 = s_2$  

$s_1 = s_2$
Decomposition Tree for SP-graphs
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Generalization: SPQR-Tree
SP-Graphs: applications

Flow Charts

PERT-Diagrams
(Program Evaluation and Review Technique)
SP-Graphs: applications

Flow Charts

Provides: Linear time algorithms for NP-complete problems (e.g., Maximum Independent Set)

PERT-Diagrams
(Program Evaluation and Review Technique)
There is a family \((G_n)_{n \in \mathbb{N}}\) of embedded SP-graphs where \(G_n\) has \(2^n\) vertices and every upward planar drawing of \(G_n\) requires \(\Omega(4^n)\) area.
There is a family $\left( G_n \right)_{n \in \mathbb{N}}$ of embedded SP-graphs where $G_n$ has $2^n$ vertices and every *upward planar drawing* of $G_n$ requires $\Omega(4^n)$ area.
Theorem [Bertolazzi et al. ’92]

There is a family \((G_n)_{n \in \mathbb{N}}\) of embedded SP-graphs where \(G_n\) has \(2^n\) vertices and every upward planar drawing of \(G_n\) requires \(\Omega(4^n)\) area.

Proof:

\[
\begin{align*}
G_0 & \quad G_n & \quad G_{n+1} \\
| & \quad | & \quad | \\
s_0 & \quad s_n & \quad s_{n+1} \\
\quad t_0 & \quad \quad t_n & \quad t_{n+1}
\end{align*}
\]
Grid Size

**Theorem** [Bertolazzi et al. '92]
There is a family \((G_n)_{n \in \mathbb{N}}\) of embedded SP-graphs where \(G_n\) has \(2^n\) vertices and every *upward planar drawing* of \(G_n\) requires \(\Omega(4^n)\) area.

**Proof:**

\[
G_0 \\
G_{n+1}
\]

\[
s_0 \\
s_{n+1}
\]

\[
t_0 \\
t_{n+1}
\]
Theorem [Bertolazzi et al. ’92]

There is a family $(G_n)_{n \in \mathbb{N}}$ of embedded SP-graphs where $G_n$ has $2^n$ vertices and every upward planar drawing of $G_n$ requires $\Omega(4^n)$ area.

Proof:

Let $G_0$ and $G_{n+1}$ be SP-graphs as described in the diagram. The proof involves showing that each step in the construction increases the area requirement by a factor of at least $4$, leading to the conclusion that the area requirement grows exponentially with $n$.

In the diagram, $s_0$ and $t_0$ are the starting points for $G_0$, while $s_n$ and $t_n$ represent the corresponding points for $G_n$. The sequence $(s_n, t_n)$ for $n \in \mathbb{N}$ illustrates the iterative construction of the SP-graphs, with $s_{n+1}$ and $t_{n+1}$ extending the sequence for $G_{n+1}$.
Grid Size

Theorem

There is a family $\left( G_n \right)_{n \in \mathbb{N}}$ of embedded SP-graphs where $G_n$ has $2^n$ vertices and every upward planar drawing of $G_n$ requires $\Omega(4^n)$ area.

Proof:

$$G_0 \quad G_{n+1}$$

$$s_0 \quad s_{n+1}$$

$$t_0 \quad t_{n+1}$$

Theorem [Bertolazzi et al. ’92]
Grid Size

**Theorem** [Bertolazzi et al. '92]

There is a family \((G_n)_{n \in \mathbb{N}}\) of embedded SP-graphs where \(G_n\) has \(2^n\) vertices and every *upward planar drawing* of \(G_n\) requires \(\Omega(4^n)\) area.

**Proof:**

\[ G_0 \]

\[ G_{n+1} \]
Theorem [Bertolazzi et al. ’92]

There is a family \((G_n)_{n \in \mathbb{N}}\) of embedded SP-graphs where \(G_n\) has \(2^n\) vertices and every \textit{upward planar drawing} of \(G_n\) requires \(\Omega(4^n)\) area.

Proof:

- For \(G_0\), \(s_0\) and \(t_0\) are placed on the x-axis.
- For \(G_n\) and \(G_{n+1}\), \(s_n\) and \(t_n\) are placed on the x-axis.
- The upward planar drawing of \(G_n\) requires \(\Omega(4^n)\) area.
Theorem [Bertolazzi et al. ’92]

There is a family \((G_n)_{n \in \mathbb{N}}\) of embedded SP-graphs where \(G_n\) has \(2^n\) vertices and every upward planar drawing of \(G_n\) requires \(\Omega(4^n)\) area.

Proof:

\[
a(G_{n+1}) \geq a(\Pi) + a(\Delta_1) + a(\Delta_2)
\]
Theorem [Bertolazzi et al. ’92]

There is a family \((G_n)_{n \in \mathbb{N}}\) of embedded SP-graphs where \(G_n\) has \(2^n\) vertices and every upward planar drawing of \(G_n\) requires \(\Omega(4^n)\) area.

Proof:

\[
\begin{align*}
 a(G_{n+1}) & \geq a(\Pi) + a(\Delta_1) + a(\Delta_2) \\
 & \geq 2 \cdot a(\Pi) \\
 & \geq 4 \cdot a(G_n)
\end{align*}
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Grid Size

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