Algorithms for Graph Visualization

Summer Semester 2017
Lecture # 5

Graph Drawing via Canonical Orders

(Partly based on lecture slides by Philipp Kindermann & Alexander Wolff)
Outline

Planar Graphs: Background

The Canonical Order of a Planar Graph

Straight-line Drawing using a Canonical Order

Geometric Representations using Canonical Orders
Planar Graphs: basics

A graph is **planar** when its vertices and edges can be mapped to points and curves in $\mathbb{R}^2$ such that the curves are non-crossing. A graph is **plane** when it is given with an **embedding** of its vertices and edges in $\mathbb{R}^2$ which certifies its planarity.

<table>
<thead>
<tr>
<th>Embeddings of $K_4$</th>
<th>Non-planar graphs $K_5$ and $K_{3,3}$.</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Embeddings" /> <img src="image2" alt="Embeddings" /> <img src="image3" alt="Embeddings" /> <img src="image4" alt="Embeddings" /></td>
<td><img src="image5" alt="Non-planar graphs" /> <img src="image6" alt="Non-planar graphs" /></td>
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</tbody>
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- How do we define the **equivalence** of planar embeddings? By the sets of **inner faces** and the **outerface**.
Characterizations, Recognition, and Drawings

1. [Kuratowski 1930: *Sur le problème des courbes gauches en topologie*] A graph is planar iff it contains neither a $K_5$ nor a $K_{3,3}$ minor.

2. [Hopcroft & Tarjan, J. ACM 1974] For a graph $G$ with $n$ vertices, there is an $O(n)$ time algorithm to test if $G$ is planar.

3. [Wagner 1936, Fáry 1948, Stein 1951] Every planar graph has an embedding where the edges are straight-lines.

4. [Koebe 1936: *Kontaktprobleme der konformen Abbildung*] Every planar graph is a circle contact graph (coin graph). (this implies 3).

![Diagram of a planar graph with coin graph representation]
1. [Tutte 1963: *How to draw a graph*]
   Every 3-connected planar has an embedding with convex polygons as its faces (i.e., implies straight-lines).
   - Idea: place vertices in the barycentre of neighbours.
   - Drawback: requires large grids.

2. [de Fraysseix, Pack, Pollack 1988]
   Every plane triangulation can be drawn with straight-lines such that the vertices reside on a \((2n - 4) \times (n - 2)\) grid.
We focus on triangulations.

- **plane triangulation** is a plane graph where every face is a triangle.
- **plane inner triangulation** is a plane graph where every face except the outer face is a triangle.

... But why??

- Easy to construct from any plane graph. Many ways to triangulate each face:

  ![Triangulation Diagram]

- Triangulations are precisely the maximal planar graphs, i.e., every planar graph is a subgraph of one such graph.
- Can we “nicely” describe all triangulations?
How to construct a plane triangulation?

- Start with a single edge $u_1u_2$. Let $G_2$ be this graph.
- Add a new vertex $u_{i+1}$ to $G_i$ so that the neighbours of $u_{i+1}$ are on the outerface of $G_i$. Let $G_{i+1}$ be this new graph.

1. Is $G_i$ a triangulation?
   - No, the neighbours of $u_{i+1}$ need to be a path.
   - No, $u_{i+1}$ also needs at least two neighbours in $G_i$.
   - No, the last vertex $v_n$ needs to cover the outerface of $G_{n-1}$.
   - Yes!

2. Do we get all plane triangulations?
   - Yes! But how can we prove this? (first we formalize the canonical order)
Canonical Order

A **canonical order** is a permutation $v_1, \ldots, v_n$ of the vertex set of a plane graph $G$ such that:

1. $v_{i+1}$ has at least two neighbours in $G_i$.
2. The neighbours of $v_{i+1}$ are consecutive in:
   
   $C_i = (v_1 = w_1, w_2, \ldots, w_{k-1}, w_k = v_2)$.

3. The neighbourhood of $v_n$ is $C_{n-1}$.
Example: How to find a Canonical Order

Idea: Start from the “last” vertex and find a “peeling” order.
Lemma: Every Plane Inner Triangulation Has a Canonical Order (CO)

- For $G = (V, E)$, proceed by induction on $|V|$.
- Base Case: $|V| = 3$ (i.e., $G$ Triangle)
- Inductive Case: $|V| > 3$, assume we have a CO for inner plane triangulations with $|V| - 1$ vertices.
- Def: A chord of $G$ is an edge connecting non-consecutive vertices on $G$'s outerface.
- Claim 1: If $G$ has a vertex $v$ on its outerface which does not belong to a chord, then $G \setminus v$ is an inner plane triangulation.
- Claim 2: $G$ has a vertex on its outerface which does not belong to a chord.
- Proof of Claim 2: The chords are nested, i.e., some chord has no chord “above” it. This “top” chord has a vertex “above” it.
- qed.
Canonical Order: Algorithm

forall the \( v \in V \) do
\[
\text{chords}(v) \leftarrow 0; \text{out}(v) \leftarrow \text{false}; \text{mark}(v) \leftarrow \text{false};
\]
\[
\text{out}(v_1), \text{out}(v_2), \text{out}(v_n) \leftarrow \text{T};
\]
for \( k = n \) to 3 do
\[
\text{pick } v \neq v_1, v_2 \text{ with } \text{mark}(v) = \text{F}, \text{out}(v) = \text{T}, \text{chords}(v) = 0;
\]
\[
v_k \leftarrow v; \text{mark}(v) \leftarrow \text{T};
\]
\[
(w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2) \leftarrow \text{Outerface}(G_{k-1});
\]
\[
(w_p, \ldots, w_q) \leftarrow \text{unmarked neighbours of } v_k;
\]
for \( i = p \) to \( q \) do \( \text{out}(w_i) \leftarrow \text{T}; \)
\[
\text{update chords(·) for } w_p, \ldots, w_q \text{ and their neighbours;}
\]

\begin{itemize}
  \item chords\( (v) \) is the number of chords incident to \( v \).
  \item mark\( (v) = \text{T} \iff v \) has been picked.
  \item out\( (v) = \text{T} \iff v \) is on the outerface of \( G_k \).
\end{itemize}

Time: \( O(n) \)
The main idea:

**Invariant:** $G_{k-1}$ has been drawn so that:

- $v_1$ is at $(0, 0)$ and $v_2$ is at $(2k - 6, 0)$.
- The outerface forms an $x$-monotone curve with slopes $\pm 1$.
Example Shift Algorithm

\[
(n-2, n-2)
\]

\[
(0, 0), (2n-4, 0)
\]
How do we define the “lower” set $L(v)$?

- Each inner node is covered exactly once.
- In $G$, this cover relation defines a rooted tree.
- In each $G_i$ ($i \in \{2, \ldots, n - 1\}$), it defines a forest where the outerface contains the “roots”.
- The trees in this forest are the “bags” shown here.

**Lemma**

*Applying the shift algorithm maintains monotone x-coordinates of the outerface.*
The Shift Method: de Fraysseix, Pach und Pollack

\(v_1, \ldots, v_n\) : a canonical order of \(G\);

for \(i = 1\) to \(n\) do \(L(v_i) \leftarrow \{v_i\}\);

\(P(v_1) \leftarrow (0, 0)\); \(P(v_2) \leftarrow (2, 0)\); \(P(v_3) \leftarrow (1, 1)\);

for \(k = 4\) to \(n\) do

Let \(w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2\) be the outerface of \(G_{k-1}\);

Let \(w_p, \ldots, w_q\) be the neighbours of \(v_k\);

for \(v \in \bigcup_{j=p+1}^{q-1} L(w_j)\) do

\[x(v) \leftarrow x(v) + 1;\]

for \(v \in \bigcup_{j=q}^{t} L(w_j)\) do

\[x(v) \leftarrow x(v) + 2;\]

\(P(v_i) \leftarrow\) intersection point of the lines with slope \(\pm 1\) from 
\(P(w_p)\) and \(P(w_q)\);

\(L(v_i) = \bigcup_{j=p+1}^{q-1} L(w_j) \cup \{v_i\}\);

Timing: \(O(n^2)\). Can we do it faster?
Linear Time Shifting

- Idea 1: To compute $x(v_k), y(v_k)$, we only need: the $y$-coordinates of $w_p$ and $w_q$ and the difference $x(w_q) - x(w_p)$.

- Idea 2: Instead of storing explicit $x$-coordinates we store certain $x$ differences.

\[
\begin{align*}
\text{Idea 1:} & \quad x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p)) \quad (1) \\
\text{Idea 2:} & \quad y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p)) \quad (2) \\
& \quad x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p)) \quad (3)
\end{align*}
\]
Linear Time Shifting

Idea 2: Instead of storing explicit $x$-coordinates we store certain $x$ differences. Namely, the edges from this “augmented” version of the cover tree.
Linear Time Shifting

To update the binary tree according to a new vertex \( v_k \)

- In the binary tree, we need the \( y(v_k) \) and the \( x \) differences from \( v_k \) to its covered neighbour \( w_{p+1} \) and to its “end” neighbours \( w_p \) and \( w_q \).
- Compute \( y(v_k) \) with (2), and \( \Delta_x(v_k, w_p) \) with (3).
- \( \Delta_x(v_k, w_q) = \Delta_x(w_p, w_q) - \Delta_x(v_k, w_p) \), and \( \Delta_x(v_k, v_{p+1}) = \Delta_x(w_p, w_{p+1}) - \Delta_x(v_k, w_p) \).
Intersection Representations of Graphs

Definition
For a collection $S$ of sets $S_1, \ldots, S_n$, the intersection graph $G(S)$ of $S$ has vertex set $S$ and edge set
\[ \{S_i S_j : i, j \in \{1, \ldots, n\}, i \neq j, \text{ and } S_i \cap S_j \neq \emptyset\}. \]
We call $S$ an intersection representation of $G(S)$.

Does every graph have an intersection representation?
Contact Representations of Graphs

A collection of interiorly disjoint objects $\mathcal{S} = \{S_1, \ldots, S_n\}$ is called a contact representation of its intersection graph $G(\mathcal{S})$.

- Some object-types: circles, line segments, triangles, rectangles, ...
- What about the domain? 2D, 3D, higher dimension, non-orientable?
- ...

Is the intersection graph of a contact representation always planar? No. Not even for planar object-types!
Which object-types can be used to represent all planar graphs?
Planar Graphs

- Contact Disk [Koebe 1936]
- Contact Triangles and T-shapes [de Fraysseix, Ossona de Mendez, Rosenstiehl 1994]
- Side Contact of 3D Boxes [Thomassen 1986]
- and many more!
Triangulating for representations

Goal: Prove that all planar graphs have a intersection/contact representation by some object-type $\mathcal{T}$.

▸ If we are given a plane graph, there are many ways to triangulate it – by adding edges or vertices. Recall, our previous triangulation picture:

▸ What is best for our goal? Adding vertices.

Lemma

*For any given object-type $\mathcal{T}$, if every planar triangulation has an intersection representation using $\mathcal{T}$-type objects, then every planar graph also can be represented using $\mathcal{T}$-type objects.*
Lemma

For any given object-type $\mathcal{T}$, if every planar triangulation has an intersection representation using $\mathcal{T}$-type objects, then every planar graph also can be represented using $\mathcal{T}$-type objects.

Proof

- Start with a planar graph $G$ and triangulate $G$ to get $G'$ by adding one dummy vertex for each face.
- Now, we have a $\mathcal{T}$-type intersection representation $R$ of $G'$.
- Remove the objects corresponding to dummy objects from $R$ and now we have $R'$ which represents precisely $G$.

The more general property we are exploiting is the fact that intersection classes of graphs are **hereditary**, i.e., closed under the taking of induced subgraphs.
T-contact and Triangle-contact Representations

Example Representations:

Idea: Use the canonical order. Notice any interesting invariant about the two representations? Did something change??

Observations:

- The base triangle or T-shape is precisely its position in the canonical order.
- The highest point is precisely the base of its cover neighbour from above.
T-contact and Triangle-contact Systems

Using the canonical order, we can generate a right-triangle contact representation. Note: we also get a T-contact representation.
partition of the internal edges into three spanning trees
every vertex has out-degree exactly one in $T_1$, $T_2$ and $T_3$
vertex rule: order of edges: entering $T_1$, leaving $T_2$, entering $T_3$, leaving $T_1$, entering $T_2$, leaving $T_3$. 
Schnyder Realizers Cont.

- 3 edge-disjoint spanning trees $T_1$, $T_2$, $T_3$ cover $G$.
- $T_1$, $T_2$, $T_3$ rooted at external vertices of $G$. 
Schnyder Realizers, Canonical Orders, and Representations

(a) $v_1 \quad v_2 \quad v_n$

(b) $v_1 \quad v_2 \quad v_n$

(c) $v_{n=8}$

(d) $v_1 \quad v_2 \quad v_n$

(e) $v_1 \quad v_2 \quad v_n$

(f) $v_1 \quad v_2 \quad v_n$

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Exercises

1. Canonical Orders:
   1.1 Can you describe a special canonical order to build precisely the maximal outerplane graphs (i.e., outerplane inner triangulations)? (hint: how many neighbours can $v_i$ have in $G_i$?)
   1.2 Can you describe a variation on the canonical order to build precisely the maximal biparitite plane graphs (i.e., every face has 4 vertices)?

2. Contact Representations:
   2.1 Show that every maximal outerplane graphs has a contact representation by: (i) rectangles; (ii) squares.
   2.2 Show that every maximal bipartite plane graph has a contact representation by: (i) rectangles; (ii) vertical and horizontal line segments.
   2.3 Show that there is a planar graph which does not have a contact representation by line segments. Note: here we do not restrict the slopes on the line segments in any way. Hint: how many edges can there be in the intersection graph of such a contact representation?