Algorithms for Graph Visualization

Summer Semester 2017

Lecture #3

Bend-minimization in Orthogonal Drawings

(based on slides from Martin Nöllenburg and Robert Görke, KIT)
Examples of graph drawing problems for planar graphs.

⇒ orthogonal drawings
Examples of graph drawing problems for planar graphs.

- orthogonal drawings
- upward drawings of acyclic graphs
Examples of graph drawing problems for planar graphs.

- orthogonal drawings
- upward drawings of acyclic graphs
- angular resolution in straight line drawings
Examples of graph drawing problems for planar graphs.

- orthogonal drawings
- upward drawings of acyclic graphs
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→ Model these problems via flow networks.
### Definition

A graph \( G = (V, E) \) is called **planar**, when there is a drawing \( \Gamma \) of \( G \) in the plane such that no pair of edges of \( G \) cross.

Such a drawing \( \Gamma \) is called a **plane embedding**.
Planarity

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**Definition (combinatorial) embedding**
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**Definition (combinatorial) embedding**

*Equivalence of drawings:*
Planarity

**Definition**

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Such a drawing \( \Gamma \) is called a **plane embedding**.

**Definition (combinatorial) embedding**

Equivalence of drawings: \( \Gamma_1 \sim \Gamma_2 \) \iff for each vertex \( v \): the edges incident to \( v \) occur in the same circular order around \( v \) in both \( \Gamma_1 \) and \( \Gamma_2 \).
Vertices, Edges, and Faces

Each drawing \( \Gamma \) of a planar graph divides the plane into faces.
Vertices, Edges, and Faces

Each drawing $\Gamma$ of a planar graph divides the plane into \textit{faces}.

The faces are the connected regions of $\mathbb{R}^2 \setminus \Gamma$. 
Vertices, Edges, and Faces

⇒ Each drawing \( \Gamma \) of a planar graph divides the plane into faces.

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⇒ \( F := \text{the faces of a planar embedding of } G = (V, E) \).
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$F :=$ the faces of a planar embedding of $G = (V, E)$.

Is there a relationship between $|V| = n$, $|E| = m$, $|F| := f$?
Vertices, Edges, and Faces

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\[ F := \text{the faces of a planar embedding of } G = (V, E). \]

Is there a relationship between \( |V| = n, |E| = m, |F| =: f \)?

Euler’s Formula:

\[ n - m + f = k + 1, \]

where \( k = \text{the number of connected components of } G. \)
Vertices, Edges, and Faces

▷ Each drawing $\Gamma$ of a planar graph divides the plane into faces.

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Is there a relationship between $|V| = n, |E| = m, |F| =: f$?

Euler’s Formula:

$$n - m + f = k + 1,$$

where $k = $ the number of connected components of $G$.

Proof?
Definition
s-t Flow

**Definition**

Given: flow network \((G = (V, E); s, t; c)\) with
$s$-$t$ Flow

**Definition**

Given: flow network $(G = (V, E); s, t; c)$ with

$\Rightarrow$ directed Graph $G = (V, E)$
s-t Flow

Definition

Given: flow network \((G = (V, E); s, t; c)\) with

- directed Graph \(G = (V, E)\)
- edge capacities \(c: E \rightarrow \mathbb{R}_{>0}\)
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- *source* \(s \in V\), *sink* \(t \in V\)
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An assignment \(f : E \rightarrow \mathbb{R}_{\geq 0}\) is an \(s-t\)-flow, when:
s-t Flow

Definition

Given: flow network \((G = (V, E); s, t; c)\) with

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\begin{align*}
\text{directed Graph } & G = (V, E) \\
\text{edge capacities } & c : E \rightarrow \mathbb{R}_{>0} \\
\text{source } & s \in V, \text{ sink } t \in V
\end{align*}
\]

An assignment \(f : E \rightarrow \mathbb{R}_{\geq 0}\) is an s-t-flow, when:

- for every \(e \in E\): \(f(e) \leq c(e)\)
- for every \(v \in V \setminus \{s, t\}\): \(\sum_{vw \in E} f(vw) - \sum_{uv \in E} f(uv) = 0\)
s-t Flow

**Definition**

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\[\text{outflow } f(v)\]
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\[\text{outflow } f(v) \quad \text{inflow } f(v)\]
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\[
\text{for every } v \in V \setminus \{s, t\}:\quad \sum_{vw \in E} f(vw) - \sum_{uv \in E} f(uv) = 0
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\[
\text{deficit}_f(v) := \quad \text{outflow } f(v) - \text{inflow } f(v)
\]
General Flow Networks

**Definition**

Given: *flow network* \((G = (V, E); l; u; b)\) with

- directed graph \(G = (V, E)\)
- edge lowerbounds \(l : E \rightarrow \mathbb{R}_{\geq 0}\)
- edge upperbounds \(u : E \rightarrow \mathbb{R}_{> 0}\)
- vertex deficits \(d : V \rightarrow \mathbb{R}\) with \(\sum_{v \in V} d(v) = 0\)
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An assignment \(X : E \to \mathbb{R}_{\geq 0}\) is a valid flow, when:

- for every \(e \in E\): \(l(e) \leq X(e) \leq u(e)\)
- for every \(v \in V\): \(\sum_{vw \in E} X(vw) - \sum_{uv \in E} X(uv) = d(v)\)
General Flow Networks

Definition

Given: flow network \((G = (V, E); l; u; b)\) with

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\geq & \text{ edge lowerbounds } l : E \rightarrow \mathbb{R}_{\geq 0} \\
\geq & \text{ edge upperbounds } u : E \rightarrow \mathbb{R}_{> 0} \\
\geq & \text{ vertex deficits } d : V \rightarrow \mathbb{R} \text{ with } \sum_{v \in V} d(v) = 0
\end{align*}\]

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\[l(e) \leq X(e) \leq u(e)\]

for every \(v \in V\):

\[
\sum_{vw \in E} X(vw) - \sum_{uv \in E} X(uv) = d(v)
\]

\[\text{deficit}_X(v)\]
Questions on flow networks

Valid flows

Find a valid flow \( X : E \rightarrow \mathbb{R}_{\geq 0} \), i.e., an edge assignment where:
\( \gg \) the bounds are satisfied, also \( l \leq X \leq u \), and
\( \gg \) the deficits/demands \( d \) are met.
Questions on flow networks

Valid flows

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$\gg$ the bounds are satisfied, also $l \leq X \leq u$, and
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Weighted version, given cost: $E \to \mathbb{R}_{\geq 0}$
Def. $\text{cost}(X) := \sum_{e \in E} \text{cost}(e) \cdot X(e)$
Questions on flow networks

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Minimum cost flow

Find a valid flow $X : E \to \mathbb{R}_{\geq 0}$, such that
- $\text{cost}(X)$ is minimized (among all valid flows).
# Questions on flow networks

**Valid flows**

Find a valid flow $X : E \rightarrow \mathbb{R}_{\geq 0}$, i.e., an edge assignment where:
- the bounds are satisfied, also $l \leq X \leq u$, and
- the deficits/demands $d$ are met.

**Weighted version, given cost**: $E \rightarrow \mathbb{R}_{\geq 0}$

Def. $\text{cost}(X) := \sum_{e \in E} \text{cost}(e) \cdot X(e)$

**Minimum cost flow**

Find a valid flow $X : E \rightarrow \mathbb{R}_{\geq 0}$, such that $\text{cost}(X)$ is minimized (among all valid flows).

**Runtime (general)**

$O(n^2 m^3 \log n)$

**Planar, edge costs $\leq c$, face size $\leq s$**

$O(c\sqrt{s} \cdot n^{3/2})$

[Cornelsen & Karrenbauer, JGAA'12]
(Planar) Orthogonal Drawing

3-Step Approach: 

\[ V = \{v_1, v_2, v_3, v_4\} \]
\[ E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}\} \]
(Planar) Orthogonal Drawing

3-Step Approach:  \textit{Topology} – \textit{Shape} – \textit{Metrics}

\[ V = \{ v_1, v_2, v_3, v_4 \} \]
\[ E = \{ \{ v_1, v_2 \}, \{ v_1, v_3 \}, \{ v_1, v_4 \}, \{ v_2, v_3 \}, \{ v_2, v_4 \} \} \]
(Planar) Orthogonal Drawing

3-Step Approach:  Topology – Shape – Metrics

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3-Step Approach: Topology – Shape – Metrics

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Problem 2: Bend minimization for fixed embeddings.

Given Graph $G = (V, E)$ with max-degree $\Delta \leq 4$, combinatorial embedding $\mathcal{F}$ and outerface $f_0$, find orthogonal grid-drawing of $(\mathcal{F}, f_0)$ minimizing bends.
The Problem

**Problem 2': Orthogonal Description**

**Given** Graph $G = (V, E)$ with max-degree $\Delta \leq 4$, combinatorial embedding $\mathcal{F}$ and outerface $f_0$, find orthogonal description $H(G)$ of $(\mathcal{F}, f_0)$ minimizing bends.
Orthogonal Description

**In:** $G = (V, E)$ planar, $\mathcal{F}$, $f_0$

**Out:** orthogonal description $H(G) = \{H(f) \mid f \in \mathcal{F}\}$

sample $G$: 

![Diagram of graph $G$ with edges and nodes labeled $f_0$, $f_1$, $f_2$, $e_1$, $e_2$, $e_3$, $e_4$, $e_5$, and $e_6$.]
Orthogonal Description

**In:** \( G = (V, E) \) planar, \( F, f_0 \)

**Out:** orthogonal description \( H(G) = \{ H(f) \mid f \in F \} \)

**Face description** \( H(f) \): edges in clockwise order. Each edge \( e \) comes with \( \delta \), and \( \alpha \). (def. as follows):

\[
H(G) :
-
H(f_1) = ((e_1, 00, \frac{3\pi}{2}),

sample \ G :
Orthogonal Description

**In:** $G = (V, E)$ planar, $\mathcal{F}, f_0$

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**Face description $H(f)$:** edges in clockwise order.
Each $e$ comes with $\delta$, and $\alpha$. (def. as follows):

$\Rightarrow \delta$ is 0-1-sequence ($0 =$ right turn, $1 =$ left turn)

**$H(G)$:**

$H(f_1) = ((e_1, 00, \frac{3\pi}{2})$, }

**Sample $G$:**

![Sample Graph](image)
Orthogonal Description

**In:** $G = (V, E)$ planar, $\mathcal{F}$, $f_0$

**Out:** orthogonal description $H(G) = \{H(f) \mid f \in \mathcal{F}\}$

**Face description $H(f)$:** edges in clockwise order. Each $e$ comes with $\delta$, and $\alpha$. (def. as follows):

- $\delta$ is 0-1-sequence ($0 =$ right turn, $1 =$ left turn)
- $\alpha$ is the angle $\in \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ between $e$ and the next edge $e'$ around $f$.

**$H(G)$:**

$H(f_1) = ((e_1, 00, \frac{3\pi}{2}),$
Orthogonal Description

In: \( G = (V, E) \) planar, \( F, f_0 \)
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&\Rightarrow \delta \text{ is 0-1-sequence (0 = right turn, 1 = left turn)} \\
&\Rightarrow \alpha \text{ is the angle } \in \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\} \text{ between } e \text{ and the next edge } e' \text{ around } f.
\end{align*}
\]

\( H(G) \):

\[
\begin{align*}
H(f_1) &= ((e_1, 00, \frac{3\pi}{2}), (e_2, \emptyset, \frac{\pi}{2}), (e_6, 00, \pi)) \\
H(f_2) &= ((e_5, 000, \frac{\pi}{2}), (e_6, 11, \frac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \frac{\pi}{2})) \\
H(f_0) &= ((e_1, 11, \frac{\pi}{2}), (e_5, 111, \frac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \frac{\pi}{2}))
\end{align*}
\]
Orthogonal Description

**In:** $G = (V, E)$ planar, $F$, $f_0$

**Out:** orthogonal description $H(G) = \{H(f) \mid f \in F\}$

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Example: Orthogonal Drawing

Task: Find an orthogonal drawing of $G$ according to the orthogonal description $H(G)$!

$H(G)$:

$H(f_1) = ((e_1, 00, \frac{3\pi}{2}), (e_2, \emptyset, \frac{\pi}{2}), (e_6, 00, \pi))$

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Example: Orthogonal Drawing

Task: Find an orthogonal drawing of $G$ according to the orthogonal description $H(G)$!

Solution:

$$H(G):$$

$$H(f_1) = ((e_1, 00, \frac{3\pi}{2}), (e_2, \emptyset, \frac{\pi}{2}), (e_6, 00, \pi))$$

$$H(f_2) = ((e_5, 000, \frac{\pi}{2}), (e_6, 11, \frac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \frac{\pi}{2}))$$

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**Valid orthogonal descriptions**

1) $H(G)$ corresponds to $(\mathcal{F}, f_0)$. 

![Diagram showing orthogonal descriptions with vertices labeled $f_0$, $e_1$, $e_2$, $e_3$, $e_4$, $e_5$, and $e_6$. Edges are labeled with angles $\pi/2$, $\pi$, and $3\pi/2$. The vertices are connected with directed edges indicating the orthogonal descriptions.]
Valid orthogonal descriptions

1) $H(G)$ corresponds to $(\mathcal{F}, f_0)$.

2) For an edge $\{u, v\}$ between faces $f$ and $g$ with $(uv, \delta_1, \alpha_1) \in H(f)$ and $(vu, \delta_2, \alpha_2) \in H(g)$, need: $\delta_1$ is inverted and reversed $\delta_2$. 
Valid orthogonal descriptions

1) $H(G)$ corresponds to $(\mathcal{F}, f_0)$.

2) For an edge $\{u, v\}$ between faces $f$ and $g$ with $(uv, \delta_1, \alpha_1) \in H(f)$ and $(vu, \delta_2, \alpha_2) \in H(g)$, need: $\delta_1$ is inverted and reversed $\delta_2$.

3) Let $|\delta|_0$ and $|\delta|_1$ be the number of zeros and ones in $\delta$. Let $r = (e, \delta, \alpha)$. For $C(r) := |\delta|_0 - |\delta|_1 + 2 - \alpha/\pi^2$:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{o.w.} \end{cases}$$
Valid orthogonal descriptions

1) $H(G)$ corresponds to $(\mathcal{F}, f_0)$.

2) For an edge \( \{u, v\} \) between faces $f$ and $g$ with $(uv, \delta_1, \alpha_1) \in H(f)$ and $(vu, \delta_2, \alpha_2) \in H(g)$, need: $\delta_1$ is inverted and reversed $\delta_2$.

3) Let $|\delta|_0$ and $|\delta|_1$ be the number of zeros and ones in $\delta$. Let $r = (e, \delta, \alpha)$. For $C(r) := |\delta|_0 - |\delta|_1 + 2 - \alpha/\pi$:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{o.w.} \end{cases}$$

4) For each node $v$ the sum of adjacent angles is $2\pi$. 
## Recall

### Problem 2’: Orthogonal Description

Given Graph $G = (V, E)$ with max-degree $\Delta \leq 4$, combinatorial embedding $\mathcal{F}$ and outerface $f_0$, find orthogonal description $H(G)$ of $(\mathcal{F}, f_0)$ minimizing bends.
Recall

**Problem 2’: Orthogonal Description**

Given Graph $G = (V, E)$ with max-degree $\Delta \leq 4$, combinatorial embedding $\mathcal{F}$ and outerface $f_0$, find orthogonal description $H(G)$ of $(\mathcal{F}, f_0)$ minimizing bends.

Idea: Build flow network!

$\Rightarrow$ value $= \angle \frac{\pi}{2}$
Recall

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Idea: Build flow network!

$\Rightarrow$ value $= \angle \frac{\pi}{2}$

$\Rightarrow$ Vertices $\rightarrow$ faces ($\# \angle \frac{\pi}{2}$ per face)
Recall

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Given Graph $G = (V, E)$ with max-degree $\Delta \leq 4$, combinatorial embedding $\mathcal{F}$ and outerface $f_0$, find orthogonal description $H(G)$ of $(\mathcal{F}, f_0)$ minimizing bends.

Idea: Build flow network!

- $\triangleright \text{value} = \angle \frac{\pi}{2}$
- $\triangleright \text{Vertices} \rightarrow \text{faces} (\# \angle \frac{\pi}{2} \text{ per face})$
- $\triangleright \text{Faces} \leftarrow \text{neighbouring faces} (\# \text{ bends toward the neighbour})$
The flow network $N(G)$

Definition flow network $N(G) = ((V \cup \mathcal{F}, A); l; u; b; \text{cost})$
The flow network $N(G)$

Definition flow network $N(G) = \((V \cup F, A); l; u; b; \text{cost}\)$

$$ A = \{(v, f)_{ee'} \in V \times F | v \text{ between edges } e, e' \text{ of } \partial f\}$$
The flow network $N(G)$

Definition flow network $N(G) = ((V \cup \mathcal{F}, A); l; u; b; cost)$

$$A = \{(v, f)_{ee'} \in V \times \mathcal{F} \mid v \text{ between edges } e, e' \text{ of } \partial f\}$$
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The flow network $N(G)$

Definition flow network $N(G) = ((V \cup \mathcal{F}, A); l; u; b; \text{cost})$

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*directed Multigraph!*
The flow network $N(G)$

Definition flow network $N(G) = ((V \cup \mathcal{F}, A); l; u; b; \text{cost})$

$$A = \{(v, f)_{ee'} \in V \times \mathcal{F} \mid v \text{ between edges } e, e' \text{ of } \partial f\}$$

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  $$\text{cost}(vf) = \begin{cases} \text{cost}(fg) = & \end{cases}$$
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\[ \text{cost}(vf) = 0 \quad \text{cost}(fg) = 1 \]

We model only the *number* of bends. Why is it enough?
Example flow network
Example flow network

Legend
Example flow network
Example flow network

Legend

\( \ell / u / \text{cost} \)

1/4/0

0/\(\infty\)/1

\(V\)

\(\mathcal{F}\)
Example flow network
Example flow network

Legend
- flow
- $4 = d$-value
- $\ell / u / \text{cost}$
- $1/4/0$
- $0/\infty/1$
- $V$
- $\mathcal{F}$
Example flow network

Legend

3  flow
4 = d-value
ℓ/u/cost
1/4/0
0/∞/1

V
F
Example flow network

Legend

3 flow
4 = $d$-value
$\ell/u/cost$
1/4/0
0/$\infty$/1
$V$
$F$
Example flow network

Legend

- **flow**
- **$4 = d$ -value**
- **$\ell/u/cost$**
- **$0/\infty/1$**

$V$  
$\mathcal{F}$
Example flow network

cost = 1
one bend! (outward)
Overview + Correctness

Theorem [Tamassia’87]

For an embedded graph \((G, \mathcal{F}, f_0)\), there is an orthogonal description \(H(G)\) with \(k\) bends

\[\iff\]
the flow network \(N(G)\) has a valid flow \(X\) of cost \(k\).
Overview + Correctness

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For an embedded graph \((G, F, f_0)\), there is an orthogonal description \(H(G)\) with \(k\) bends

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\[ \iff \text{In.: Flow network } N(G), \text{ flow } X \text{ with cost } k \]
\[ \text{Out.: orthogonal description } H(G) \]
## Overview + Correctness

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\(\iff:\) In.: Flow network \(N(G)\), flow \(X\) with cost \(k\)
Out.: orthogonal description \(H(G)\)

(H4) Total angle at each node \(= 2\pi\)
Overview + Correctness

**Theorem**

For an embedded graph \((G, \mathcal{F}, f_0)\), there is an orthogonal description \(H(G)\) with \(k\) bends

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(H2) bend order inverted and reversed on opposite sides ✓

(H4) Total angle at each node = \(2\pi\) ✓
Overview + Correctness

Theorem [Tamassia’87]

For an embedded graph $(G, \mathcal{F}, f_0)$, there is an orthogonal description $H(G)$ with $k$ bends

$\Leftrightarrow$ the flow network $N(G)$ has a valid flow $X$ of cost $k$.

$\Leftarrow$: In.: Flow network $N(G)$, flow $X$ with cost $k$

Out.: orthogonal description $H(G)$

(H1) $H(G)$ matches $\mathcal{F}$, $f_0$

(H2) bend order inverted and reversed on opposite sides

(H4) Total angle at each node $= 2\pi$

✓
Overview + Correctness

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For an embedded graph \((G, \mathcal{F}, f_0)\), there is an orthogonal description \(H(G)\) with \(k\) bends

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(H1) \(H(G)\) matches \(\mathcal{F}, f_0\)
(H2) bend order inverted and reversed on opposite sides
(H3) angle sum of \(f\): \(d(f)\) without flat angles \(= 4\)
(H4) Total angle at each node \(= 2\pi\)

\[ \checkmark \]
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Overview + Correctness

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\((N1)\ X(vf) = 1/2/3/4 \checkmark\)
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Overview + Correctness

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\((N1)\) \(X(vf) = 1/2/3/4\) ✓

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\((N3)\) capacities ✓, deficit/demand coverage
### Overview + Correctness

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\(N1\) \(X(vf) = 1/2/3/4 \checkmark\)

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\(N3\) capacities \(\checkmark\), deficit/demand coverage \(\checkmark\)

\(N4\) cost = \(k \checkmark\)
Overview + Correctness

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\(N4\) cost = \(k\) \(\checkmark\)

Runtime:
Using values \((l,u,c)\), and amount of edges,...

\[ O(n^2 \log n) \quad [\text{Tamassia '87}] \]
\[ O(n^{7/4} \sqrt{\log n}) \quad [\text{Garg & Tamassia '96}] \]
\[ O(n^{3/2}) \quad [\text{Cornelsen & Karrenbauer '12}] \]
Compactifying

Problem: orthogonal drawing from an orthogonal description

Given planar graph $G = (V, E)$ with $\Delta \leq 4$ and an orthogonal description $H(G)$.

Find an orthogonal drawing of $G$, realizing $H(G)$. 
Compactifying

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Special Case: all faces are rectangles

⇒ guarantees: 
  - minimum total edge length
  - minimum area
Compactifying

Problem: orthogonal drawing from an orthogonal description

Given planar graph $G = (V, E)$ with $\Delta \leq 4$ and an orthogonal description $H(G)$.
Find an orthogonal drawing of $G$, realizing $H(G)$.

Special Case: all faces are rectangles
⇒ guarantees: ≫ minimum total edge length
  ≫ minimum area
  ≫ bends are on the outer face
  ≫ opposite sides have equal length ⇒ Layout ok
Flow network length assignment

Definition flow network $N_{\text{hor}} = ((V_{\text{hor}}, A_{\text{hor}}); l; u; b; \text{cost})$

- $V_{\text{hor}} = \mathcal{F}$
- $A_{\text{hor}} = \{(f, g) \mid f, g \text{ have a common horizontal line segment and } f \text{ is below } g\}$
- $l \equiv 1$
- $u \equiv \infty$
- $\text{cost} \equiv 1$
- $d \equiv 0$
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Definition flow network \( N_{\text{hor}} = ((V_{\text{hor}}, A_{\text{hor}}); l; u; b; \text{cost}) \)

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\[
\begin{align*}
    V_{\text{hor}} &= \mathcal{F} \\
    A_{\text{hor}} &= \{(f, g) \mid f, g \text{ have a common horizontal line segment and } f \text{ is below } g\} \\
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Flow network length assignment

**Definition flow network** $N_{\text{ver}} = ((V_{\text{ver}}, A_{\text{ver}}); l; u; b; \text{cost})$

- $V_{\text{ver}} = F$
- $A_{\text{ver}} = \{(f, g) \mid f, g \text{ have a common vertical line segment and } f \text{ is left of } g\}$
- $l \equiv 1$
- $u \equiv \infty$
- $\text{cost} \equiv 1$
- $d \equiv 0$
Flow network length assignment

**Definition flow network** $N_{ver} = ((V_{ver}, A_{ver}); l; u; b; cost)$

\[ V_{ver} = \mathcal{F} \]
\[ A_{ver} = \{(f, g) \mid f, g \text{ have a common vertical line segment and } f \text{ is left of } g \} \]
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- $l \equiv 1$
- $u \equiv \infty$
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- $d \equiv 0$
Optimal with Rectangles

**Theorem**

Min-Cost-flow for $N_{\text{hor}}$ and $N_{\text{ver}}$ implies:

1. $X$ valid flow $\Leftrightarrow$ corr. edge lengths induce orth. drawing
2. $|X(N_{\text{hor}})| = \text{width}, |X(N_{\text{ver}})| = \text{height}$
3. cost($X(N_{\text{hor}})$) + cost($X(N_{\text{ver}})$) = total edge length
Refinement of \((G, H)\) – inner face
Refinement of \((G, H)\) – inner face

\[
\begin{align*}
&\text{corner}(e) \\
&\text{next}(e)
\end{align*}
\]
Refinement of \((G, H)\) – inner face
Refinement of \((G, H)\) – inner face

\[ e_0 \quad e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \quad e_6 \quad e_7 \quad e_8 \quad e_9 \quad e_{10} \quad e_{11} \quad e_{12} \quad e_{13} \quad e_{14} \quad e_{15} \]

\[ \text{front}(e_0) \]

\(f\)
Refinement of \((G, H)\) – inner face

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Refinement of \((G, H)\) – inner face
Refinement of \((G, H)\) – outerface

![Diagram of a graph with labeled edges]
Refinement of \((G, H) – \text{outerface}\)
Refinement of \((G, H)\) – outerface
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Area minimized?
Refinement of \((G, H)\) – outerface

Area minimized?  
No!
Refinement of \((G, H)\) – outerface

Area minimized?

No!

But …. do we get some bounds on the area?
Refinement of \((G, H)\) – outerface

Area minimized?

No!

But …. do we get some bounds on the area?

Yes! \(O((n + b)^2)\)
Summary Bend-minimization

- bend-minimization for fixed embeddings via flow networks.
- optimal compactifying for rectangular faces
- extension to the general faces is not optimal :( 
- compactification is NP-hard !!
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Possible Extensions:
- degree $\geq 4$
- non-planar graphs

exercise
Summary Bend-minimization

- bend-minimization for fixed embeddings via flow networks.
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- degree $\geq 4$
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exercise
Compactifying is NP-hard [Patrignani ’01]

⇒ Reduction via SAT
Compactifying is NP-hard \cite{Patrignani ’01}

\begin{itemize}
\item Reduction via **SAT**
\item $n$ variables $x_1, \ldots, x_n$
\item $m$ clauses $C_1, \ldots, C_m$;
\item each clause: Disjunction of literals $x_i \lor \overline{x_i}$
\quad e.g.: $C_1 = x_1 \lor \overline{x_2} \lor x_3$
\item Is $\Phi = C_1 \land C_2 \land \ldots \land C_m$ satisfiable, i.e., is there an assignment to the variables satisfying every clause?
\end{itemize}
Compactifying is NP-hard \cite{Patrignani '01}

Reduction via \textbf{SAT}

- \( n \) variables \( x_1, \ldots, x_n \)
- \( m \) clauses \( C_1, \ldots, C_m \);
- each clause: Disjunction of literals \( x_i/x_i \)
  
  e.g.: \( C_1 = x_1 \lor \overline{x_2} \lor x_3 \)

Is \( \Phi = C_1 \land C_2 \land \ldots \land C_m \) satisfiable, i.e., is there an assignment to the variables satisfying every clause?

Find an appropriate value \( K \) such that \((G, H)\) can be drawn in \( K \) area \iff \( \Phi \) is satisfiable.
Compactifying is NP-hard [Patrignani ’01]

- High level structure of \((G, H)\)
  - boundary.
  - belts, and pistons
  - clause gadgets.
  - variable gadgets.
Boundary, belt, and “piston” gadget

(w × h)-rectangle
Boundary, belt, and “piston” gadget
Boundary, belt, and “piston” gadget
Boundary, belt, and “piston” gadget
Boundary, belt, and “piston” gadget
Boundary, belt, and “piston” gadget
Boundary, belt, and “piston” gadget
Boundary, belt, and “piston” gadget
Clause Gadgets

\[ C_1 \]
\[ C_2 \]
\[ C_3 \]
\[ C_4 \]

\[ x_1 \]
\[ false \]
\[ x_2 \]
\[ false \]
\[ x_3 \]
\[ true \]
\[ x_4 \]
\[ true \]
\[ x_5 \]
\[ true \]
Clause Gadgets

Example:

\[ C_1 = x_2 \lor \overline{x_4} \]
\[ C_2 = x_1 \lor x_2 \lor \overline{x_3} \]
\[ C_3 = x_5 \]
\[ C_4 = x_4 \lor \overline{x_5} \]

\[ C_1 = x_2 \lor \overline{x_4} \]
\[ C_2 = x_1 \lor x_2 \lor \overline{x_3} \]
\[ C_3 = x_5 \]
\[ C_4 = x_4 \lor \overline{x_5} \]
Clause Gadgets

Example:

\[
C_1 = x_2 \lor \overline{x_4} \\
C_2 = x_1 \lor x_2 \lor \overline{x_3} \\
C_3 = x_5 \\
C_4 = x_4 \lor \overline{x_5}
\]
Clause Gadgets

Example:

\( C_1 = x_2 \lor \overline{x_4} \)

\( C_2 = x_1 \lor x_2 \lor \overline{x_3} \)

\( C_3 = x_5 \)

\( C_4 = x_4 \lor \overline{x_5} \)

Insert (2ⁿ − 1)-chain through each clause
Clause Gadgets

Example:

\[ C_1 = x_2 \lor \overline{x_4} \]
\[ C_2 = x_1 \lor x_2 \lor \overline{x_3} \]
\[ C_3 = x_5 \]
\[ C_4 = x_4 \lor \overline{x_5} \]

insert \((2n - 1)\)-chain through each clause
Clause Gadgets

Example:

\begin{align*}
C_1 &= x_2 \lor \overline{x_4} \\
C_2 &= x_1 \lor x_2 \lor \overline{x_3} \\
C_3 &= x_5 \\
C_4 &= x_4 \lor \overline{x_5}
\end{align*}

insert \((2n-1)\)-chain through each clause
Complete Reduction

\[ 9m + 7 \]

\[ 9n + 2 \]
Complete Reduction

Pick

\[ K = (9n + 2) \cdot (9m + 7) \]
Complete Reduction

Pick

\[ K = (9n + 2) \cdot (9m + 7) \]

Then:

\((G, H)\) has an area \(K\) drawing

\[ \iff \Phi \text{ satisfiable} \]