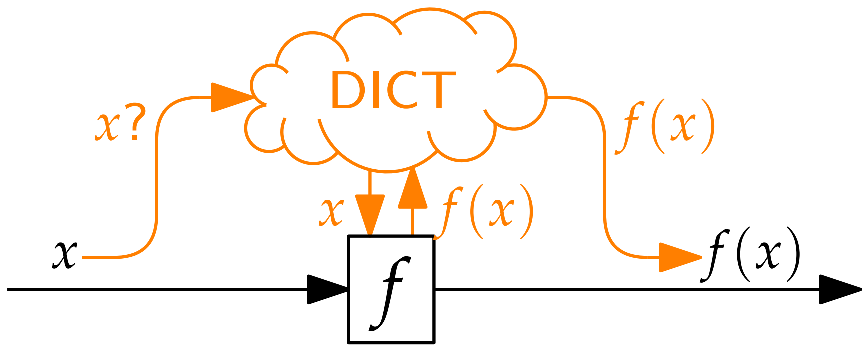
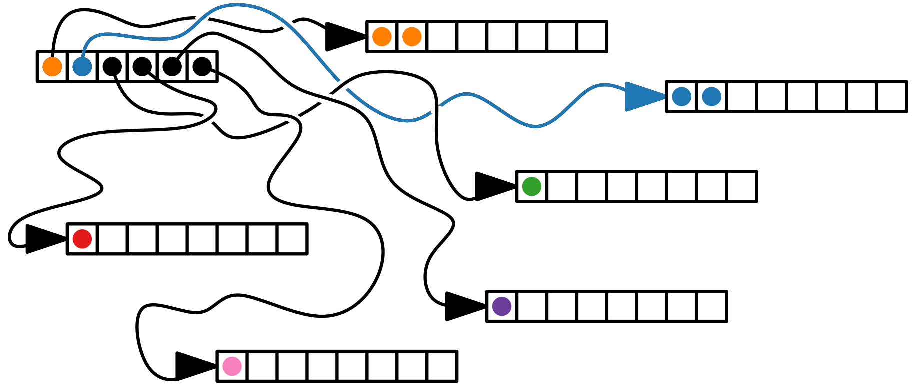
Advanced Algorithms

Algorithms in Practice

Computers are Fast · Knapsack DP · IntroSort

Tim Hegemann · WS22





Theory & Practice

Theorem. Sorting an array of comparable objects (or primitives) on the Java Virtual Machine can be solved in $\mathcal{O}(1)$ time.

Arrays on the JVM are indexed by 32-bit signed integers. So, an array cannot have more than $2^{31} - 1$ entries.

Observation. In practice, the big O notation has some shortcomings...

Remember:

$$O(g) = \left\{ f\colon \mathbb{N} \to \mathbb{R} \;\middle|\; \begin{array}{c} \text{there are positive constants } c \text{ and } n_0 \\ \text{such that for all } n \geq n_0\text{:} \\ f(n) \leq c \cdot g(n) \end{array} \right\}$$

What if?

- We have lots of small instances but n_0 is large?
- Two algorithms have similar g (e.g. n and $n\log(n)$) but for one c is much higher?

“In theory, there is no difference between
theory and practice;
but, in practice, there is.”

Do We Need More Detail?

[Knuth 1998]

	Step	Operations	Time
	$N3$	`CMPA, JG, JE`	$3.5u$
Either	$N4$	`STA, INC`	$3u$
	$N5$	`INC, LDA, CMPA, JGE`	$6u$
Or	$N8$	`STX, INC`	$3u$
	$N9$	`DEC, LDX, CMPX, JGE`	$6u$

Thus about $12.5u$ is spent on each record in each pass, and the total running time will be asymptotically $12.5N \lg N$, for both the average case and the worst case. This is slower than quicksort's average time, and it may not be enough better than heapsort to justify taking twice as much memory space, since the asymptotic running time of Program 5.2.3H is never more than $18N \lg N$.

What about?

- Caching?
- Super-scalar Architectures?
- Out-of-order Processors?
- Vector Processors? Parallel Multiprocessors?
- Fixed-function Units?

Computers are Fast

Event	Latency
1 CPU cycle	0.3 ns
Level 1 cache access	0.9 ns
Level 2 cache access	3 ns
Level 3 cache access	10 ns
RAM access	100 ns
SSD I/O	10–100 μs
HDD I/O	1–10 ms
Internet (SF–NYC)	40 ms

https://computers-are-fast.github.io/

[Brendan Gregg. "Systems Performance: Enterprise and the Cloud." 2nd Edition 2020]

Julius-Maximilians-
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Algorithms in Practice

Implementing the Knapsack Dynamic Program

Tim Hegemann · WS22

Knapsack

Input: Finite set U, for each $u \in U$ a size $s(u) \in \mathbb{Z}^+$ and a value $v(u) \in \mathbb{Z}^+$, and positive integers B and K.

Task: Is there a subset $U' \subseteq U$ such that $\sum_{u \in U'} s(u) \leq B$ and such that $\sum_{u \in U'} v(u) \geq K$?

[Garey, Johnson 1979; Karp 1972]

Optimization Problem: Maximize K

Example: $\boxed{\$\,10 \mid △\,4}$ $\boxed{\$\,12 \mid △\,6}$ $\boxed{\$\,4 \mid △\,1}$ $\boxed{\$\,6 \mid △\,2}$, $B = 6$

Strategies:

	$\sum_{u \in U'} s(u)$	$\sum_{u \in U'} v(u)$
Greedy	6	12
Size/Weight-Ratio	3	10
OPT	6	16

Knapsack is (weakly) NP-complete but can be solved in pseudo-polynomial time by dynamic programming.

A Dynamic Program for KNAPSACK

Pick an arbitrary order on the items U.

$f(i, s)$ = max value with size limit s using only items $1 \ldots i$.

Base case:

$$f(1, s) = \begin{cases} v(1) & \text{if } s(1) \leq s \\ 0 & \text{otherwise} \end{cases}$$

Recursion:

$$f(i, s) = \max \begin{cases} f(i-1, s) \\ v(i) + f(i-1, s - s(i)) & \text{if } s \geq s(i) \end{cases}$$

Result: $f(|U|, B)$

Runtime: $\mathcal{O}(|U| \cdot B) \longleftarrow$ *"pseudo-polynomial"*

Implement it

```scala
case class Item(weight: Int, value: Int)

def solve(items: IndexedSeq[Item], weight: Int) =
  def go(i: Int, weight: Int): Int =
    val item = items(i)
    if i == 0 then
      if item.weight <= weight then item.value else 0
    else if item.weight <= weight then
      go(i - 1, weight) max
          (go(i - 1, weight - item.weight) + item.value)
    else go(i - 1, weight)

  go(items.size - 1, weight)
```

Expectation Management

[Skiena 2012]

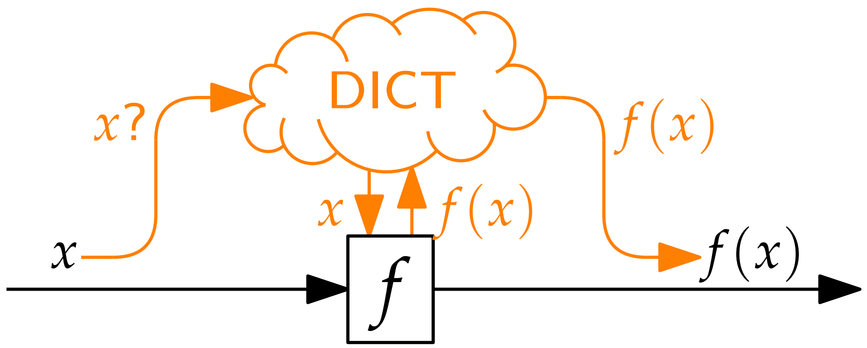
n $f(n)$	$\lg n$	n	$n \lg n$	n^2	2^n	$n!$
10	0.003 μs	0.01 μs	0.033 μs	0.1 μs	1 μs	3.63 ms
20	0.004 μs	0.02 μs	0.086 μs	0.4 μs	1 ms	77.1 years
30	0.005 μs	0.03 μs	0.147 μs	0.9 μs	1 sec	8.4×10^{15} yrs
40	0.005 μs	0.04 μs	0.213 μs	1.6 μs	18.3 min	
50	0.006 μs	0.05 μs	0.282 μs	2.5 μs	13 days	
100	0.007 μs	0.1 μs	0.644 μs	10 μs	4×10^{13} yrs	
1,000	0.010 μs	1.00 μs	9.966 μs	1 ms		
10,000	0.013 μs	10 μs	130 μs	100 ms		
100,000	0.017 μs	0.10 ms	1.67 ms	10 sec		
1,000,000	0.020 μs	1 ms	19.93 ms	16.7 min		
10,000,000	0.023 μs	0.01 sec	0.23 sec	1.16 days		
100,000,000	0.027 μs	0.10 sec	2.66 sec	115.7 days		
1,000,000,000	0.030 μs	1 sec	29.90 sec	31.7 years		

Figure 2.4: Growth rates of common functions measured in nanoseconds

Running Times

$\lvert U \rvert =$	30	100	$B = 10\lvert U \rvert$
Scala naïve	1.15 s	$\approx 2\,700 \times$ age of the universe	

Memoization




```scala
def momoized[K, V](f: K ⇒ V): K ⇒ V =
  val dict = mutable.Map.empty[K, V]
  (key: K) ⇒ dict.getOrElseUpdate(key, f(key))


lazy val go: ((Int, Int)) ⇒ Int =
  memoized((args: (Int, Int)) ⇒
    val (i, weight) = args
    // fill in the body of the old go
  )
```

Running Times

$\lvert U\rvert =$	30	100	1K	$B = 10\lvert U\rvert$
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	…	

Running Times

```
Exception in thread "main" java.lang.StackOverflowError
    at scala.runtime.BoxesRunTime.equalsNumNum(BoxesRunTime.java:148)
    at scala.runtime.BoxesRunTime.equalsNumObject(BoxesRunTime.java:138)
    at scala.runtime.BoxesRunTime.equals2(BoxesRunTime.java:127)
    at scala.runtime.BoxesRunTime.equals(BoxesRunTime.java:119)
    at scala.Tuple2.equals(Tuple2.scala:24)
    at scala.runtime.BoxesRunTime.equals2(BoxesRunTime.java:133)
    at scala.runtime.BoxesRunTime.equals(BoxesRunTime.java:119)
    at scala.collection.mutable.HashMap$Node.findNode(HashMap.scala:621)
    at scala.collection.mutable.HashMap.getOrElseUpdate(HashMap.scala:449)
    at y.Memoized$.momoized$$anonfun$1(Y.scala:64)
    at y.Memoized$.go$lzyINIT1$1$$anonfun$1(Y.scala:72)
    at y.Memoized$.momoized$$anonfun$1$$anonfun$1(Y.scala:64)
    at scala.collection.mutable.HashMap.getOrElseUpdate(HashMap.scala:454)
    at y.Memoized$.momoized$$anonfun$1(Y.scala:64)
    ...
```

Running Times

$\lvert U\rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	. . .

$B = 10\lvert U\rvert$

```
Exception in thread "main" java.lang.OutOfMemoryError: Java heap s
    at java.base/java.lang.Integer.valueOf(Integer.java:1077)
    at scala.runtime.BoxesRunTime.boxToInteger(BoxesRunTime.java:6
    at y.TCO$.go$3(Y.scala:112)
    at y.TCO$.solve(Y.scala:121)
    at y.TCO$.runTco(Y.scala:82)
    at y.runTco.main(Y.scala:80)
```

Covered in this talk:

Reducing the Memory Footprint

Idea:

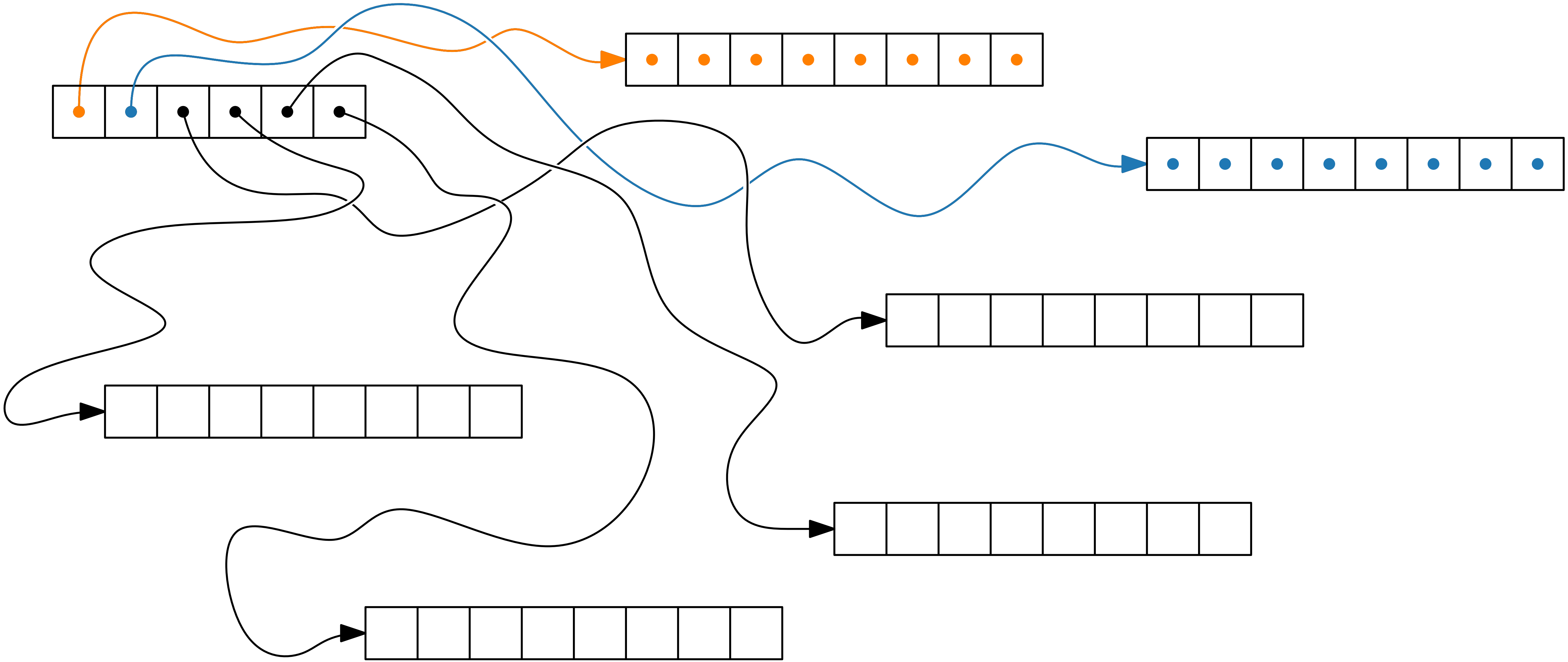
- Use Itearation instead of Recursion
- Use Arrays instead of a Dictionary

Reducing the Memory Footprint

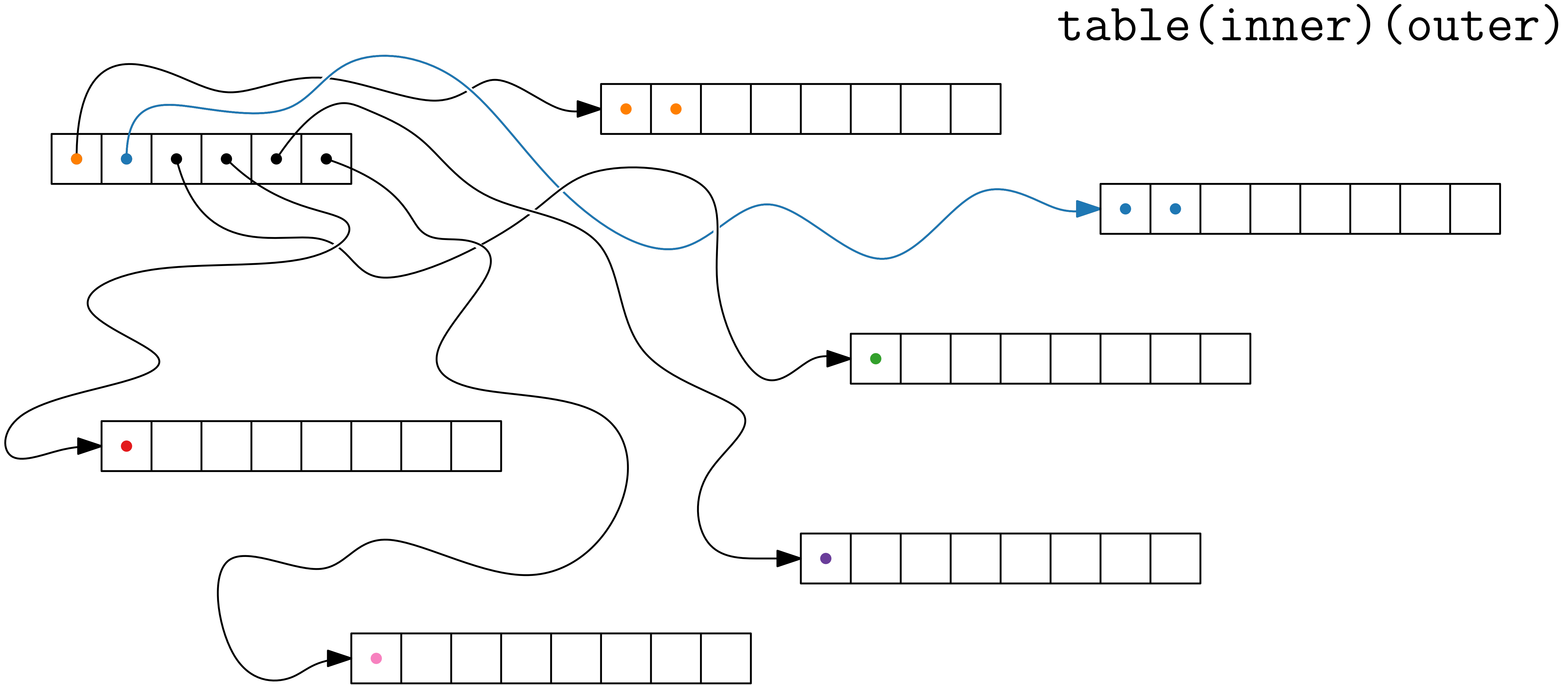
```
def solve(items: Seq[Item], weight: Int) =
  val dict = Array.fill(items.size, weight + 1)(0)
  val head = items.head
  for w ← head.weight to weight do dict(0)(w) = head.value
  for
    (item, i) ← items.zipWithIndex.tail
    w ← 0 to weight
  do
    val dontTake = dict(i - 1)(w)
    if w ⩾ item.weight then
      val take = dict(i - 1)(w - item.weight) + item.value
      dict(i)(w) = dontTake max take
    else dict(i)(w) = dontTake
  dict(items.size - 1)(weight)
```

Arrays and Memory Locality

`table(outer)(inner)`



Arrays and Memory Locality

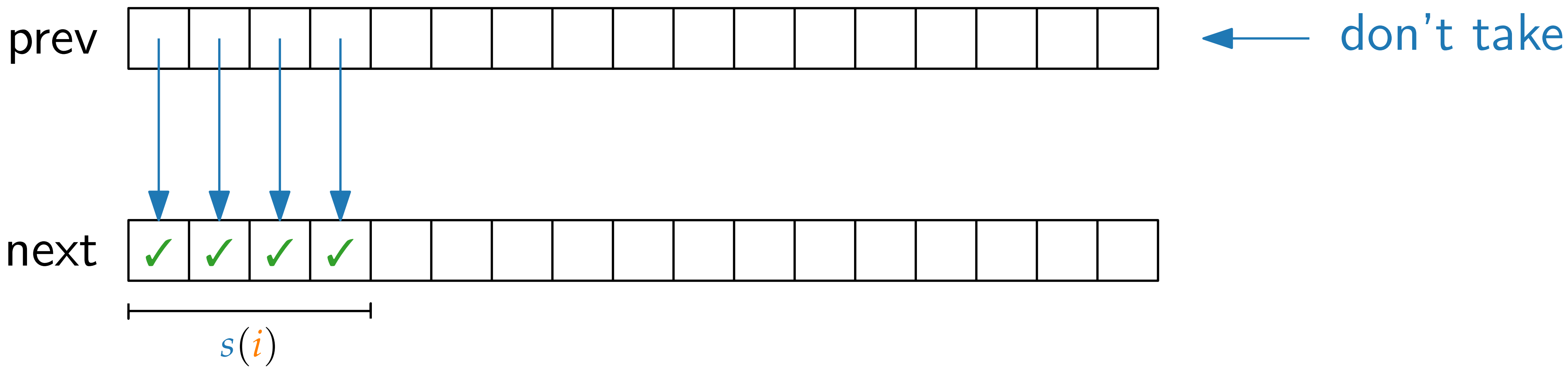


Running Times

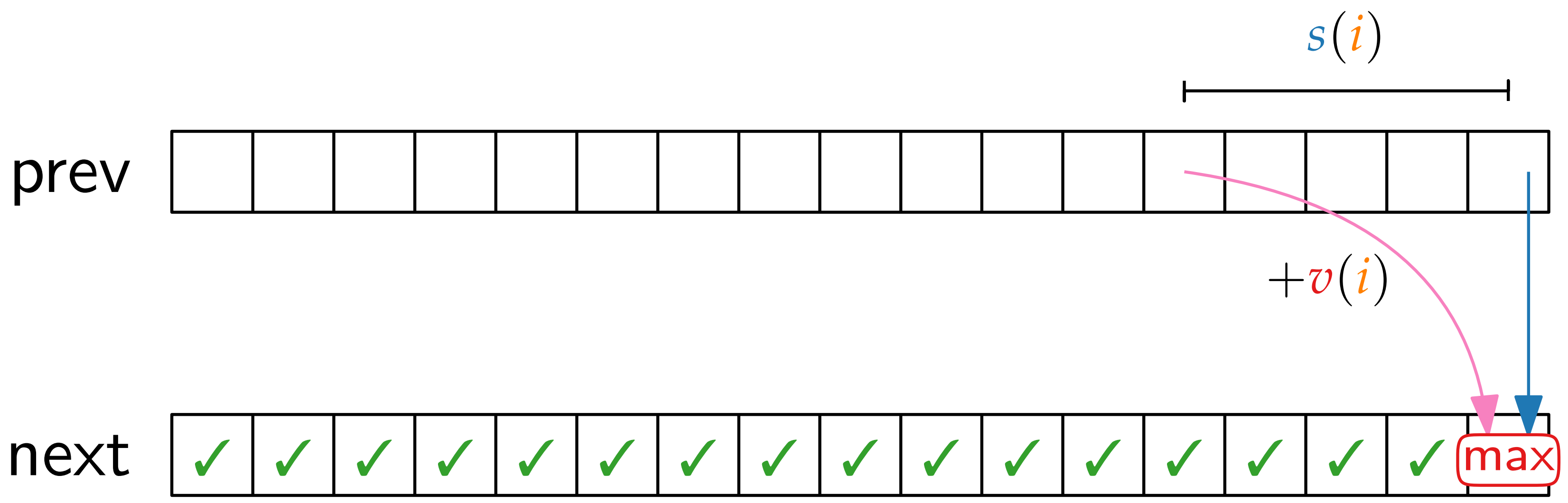
$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s

$B = 10\lvert U \rvert$

$$f(i, s) = \max \begin{cases} f(i-1, s) & \longleftarrow \text{don't take} \\ v(i) + f(i-1, s - s(i)) & \text{if } s \geq s(i) \end{cases}$$



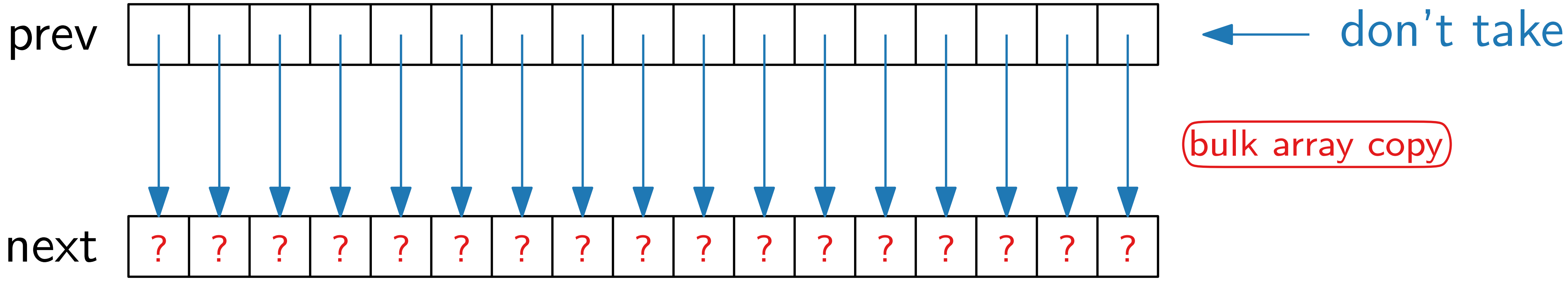
$$f(i, s) = \max \begin{cases} f(i-1, s) \\ v(i) + f(i-1, s - s(i)) & \text{if } s \geq s(i) \end{cases}$$



$$f(i,s) = \max \begin{cases} f(i-1,s) \\ v(i) + f(i-1, s-s(i)) & \text{if} \end{cases}$$

Hint:

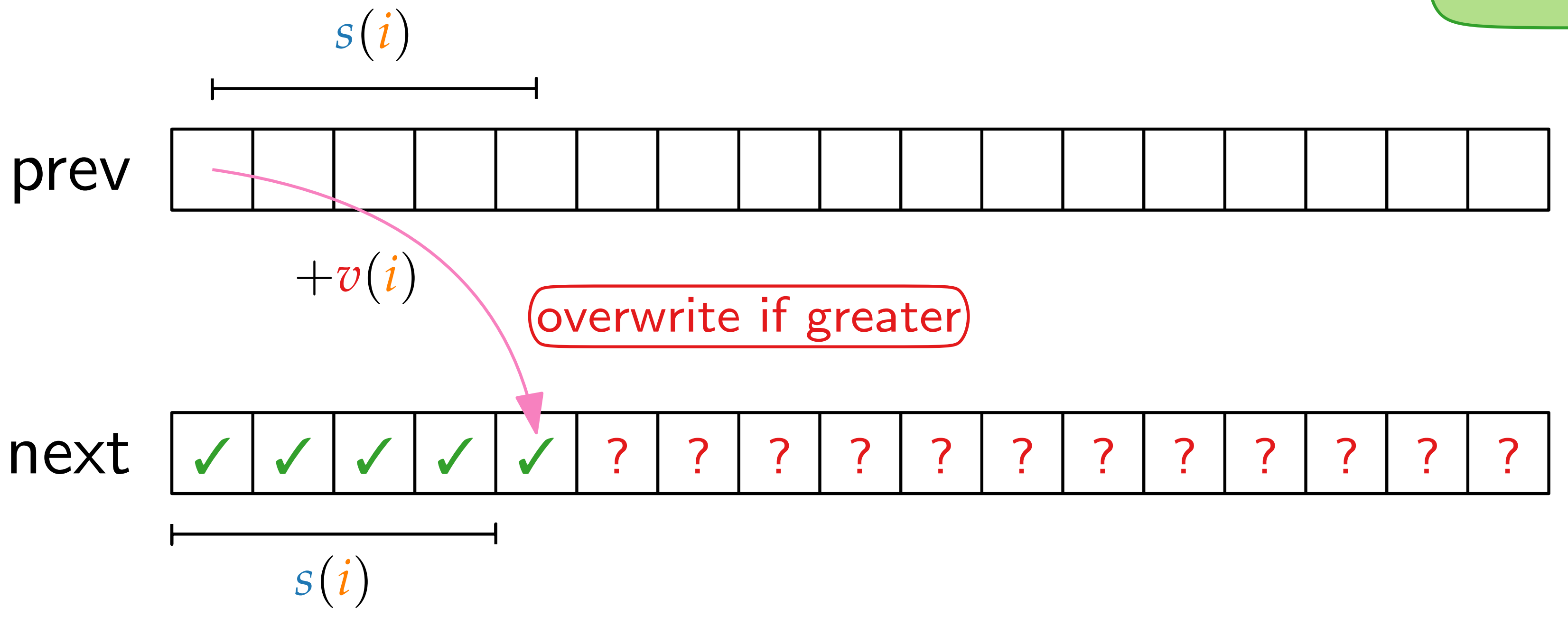
Use optimized procedures provided by your operating system!



$$f(i, s) = \max \begin{cases} f(i-1, s) & \\ v(i) + f(i-1, s - s(i)) & \text{if} \end{cases}$$

Hint:

Use optimized procedures provided by your operating system!



Running Times

$B = 10|U|$

$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				636 ms
node.js				957 ms
Python3				96.8 s
Pypy3				2.48 s
Rust (safe)				300 ms
C++ (by tvd)				212 ms
Rust with AVX2				43 ms

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How to Sort a Million 32-bit Integers

Tim Hegemann · WS22

How to Sort a Million 32-bit Integers?

Well-known comparison-based sorting algorithms

	best-case	average	worst-case	in-place
Insertion sort	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^2)$	✓
Heap sort	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n \log(n))$	✓
Quicksort	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n^2)$ **!**	✓
~~Merge sort~~	~~$\mathcal{O}(n \log(n))$~~	~~$\mathcal{O}(n \log(n))$~~	~~$\mathcal{O}(n \log(n))$~~	~~✗~~

← Very low constant factors

Observation.

- Equal integers are undistinguishable.
- Integer comparisons are probably much cheaper than copying.

What about sorting in linear time?

~~Counting sort: $\mathcal{O}(n + k)$~~

Radix sort: $\mathcal{O}(w \cdot n)$ $\quad w = 32 \quad (\log_2(10^6) \approx 20)$

Quicksort With an Emergency Exit

Observation. Qicksort has optimal runtime as long as the recursion depth is $\mathcal{O}(\log(n))$.

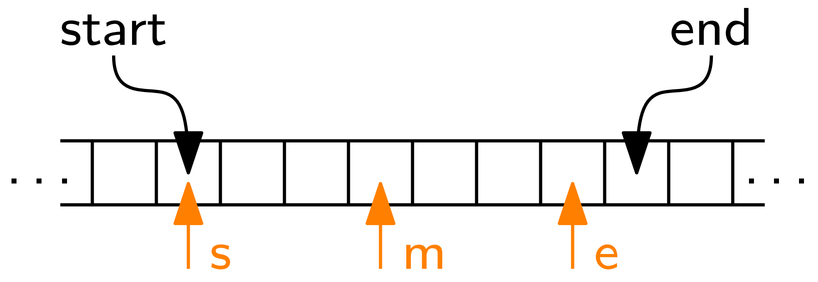
Idea. If for any partition the recursion reaches a depth of $a \log(n)$ (for a given parameter a) stop and sort this section with heap sort.

$a = 2$

INTROSORT [Musser 1997]

Common optimizations

- Median-of-3 pivoting
- Adaptive sorting: $b = 16$
  If the partition is smaller than b (for a given parameter b) stop the recursion. At the end call insertion sort to finish the mostly sorted array.



pivot $=$ median(s, m, e)

see (and hear) it!