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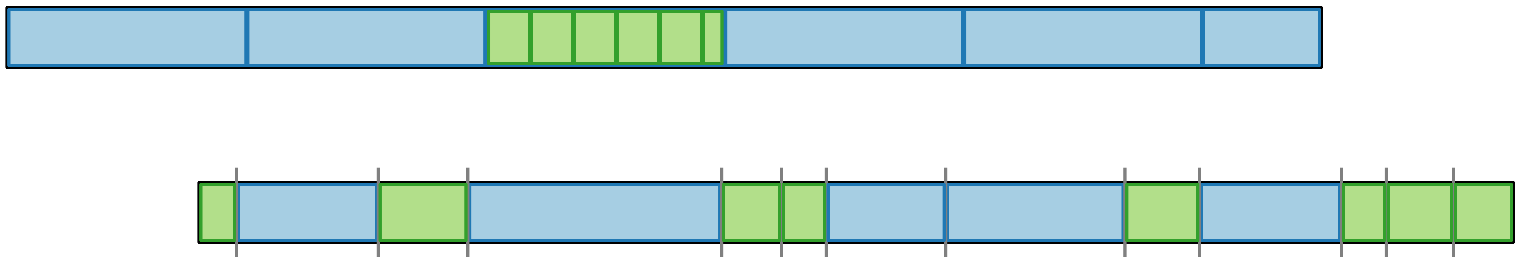
UNIVERSITÄT
WÜRZBURG

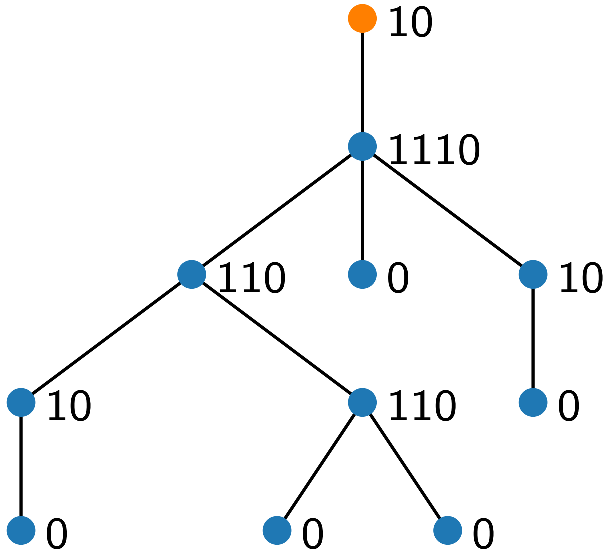
Advanced Algorithms

Succinct Data Structures

Indexable Dictionaries and Trees

Jonathan Klawitter · WS20





Data structures

A **data structure** is a concept to

- **store**,
- **organize**, and
- **manage** data.

As such, it is a collection of

- **data values**,
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- the **operations** that can applied to the data.

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Remarks.

- We look at data structures as a designer/implementer (and not necessarily as a user).
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- the **operations** that can applied to the data.

$\Rightarrow$

- What do we represent?
- How much space is required?
- Dynamic or static?
- Which operations are defined?
- How fast are they?

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Goal.

- Use space “close” to information-theoretical minimum,
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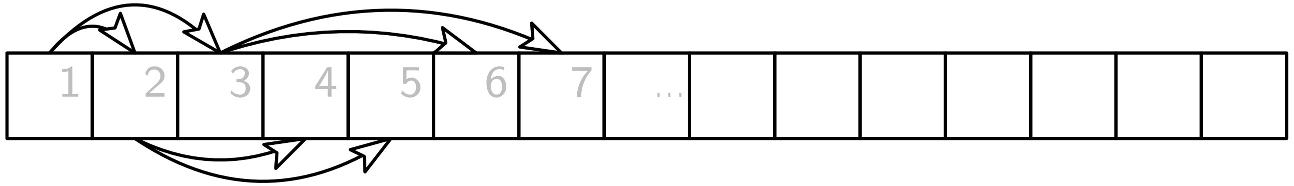

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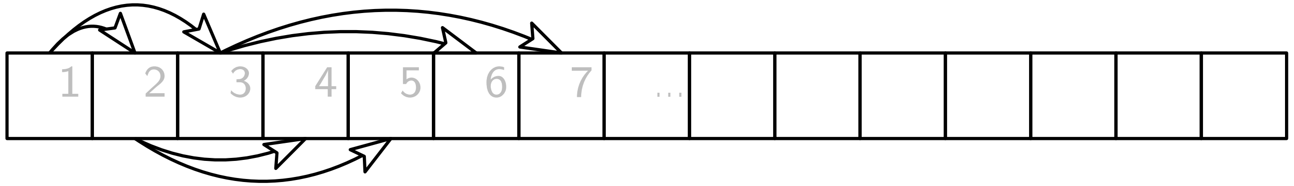

$$\texttt{leftChild}(i) = 2i$$

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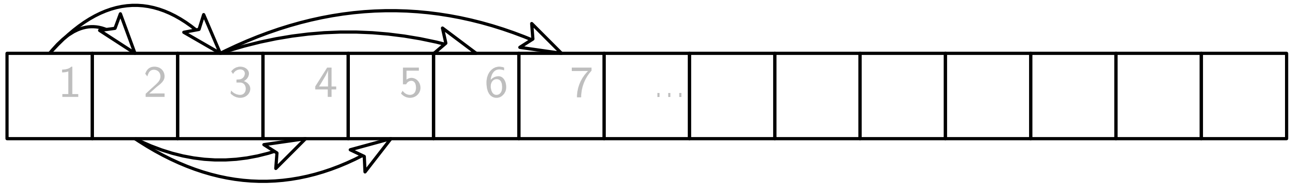

$\texttt{leftChild}(i) = 2i$

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And unbalanced trees?

Succinct indexable dictionary

Represent a subset $S \subset [n]$ and support $O(1)$-time operations:

- $\texttt{member}(i)$ returns if $i \in S$
- $\texttt{rank}(i) = \#$ 1's at or before position i
- $\texttt{select}(j) =$ position of jth 1 bit
- `predecessor` and `successor` can be answered using `rank` and `select`

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How many bits of space do we need to distinguish them?

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$$\log 2^n = n \text{ bits}$$

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	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
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$\Rightarrow$ Exercise: Use them to answer `predecessor` and `successor`.

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   and store cumulative rank: each $\log n$ bits

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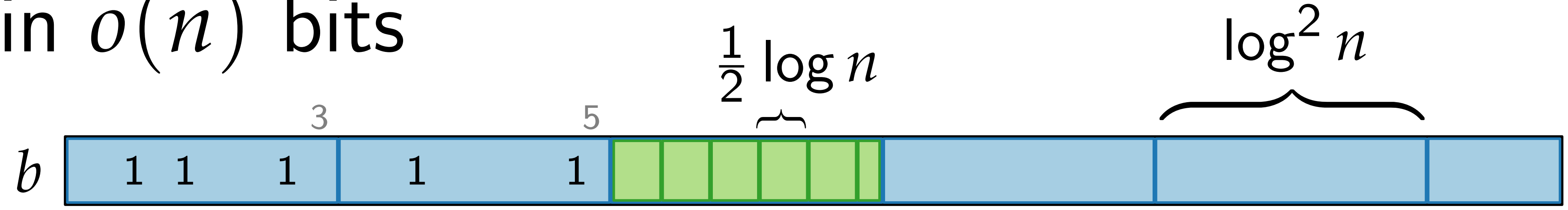
Rank in $o(n)$ bits



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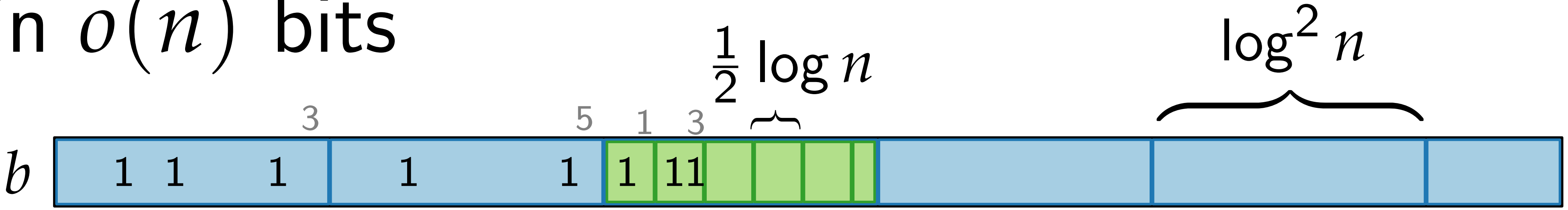

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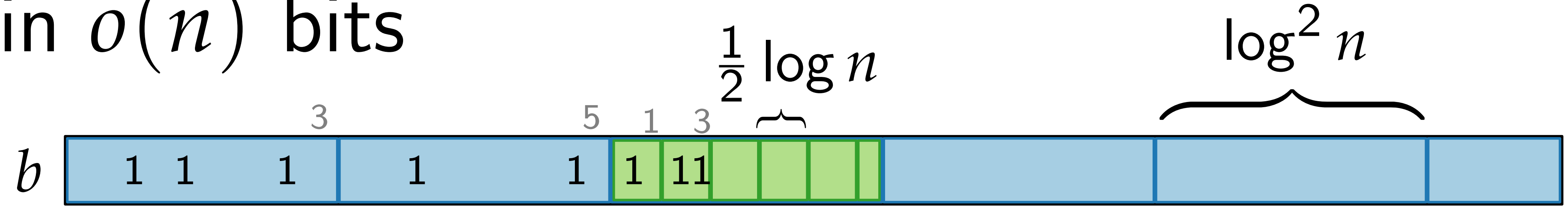

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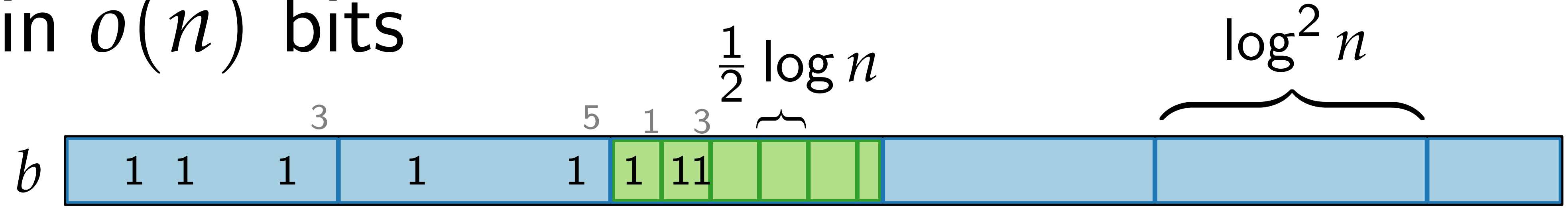

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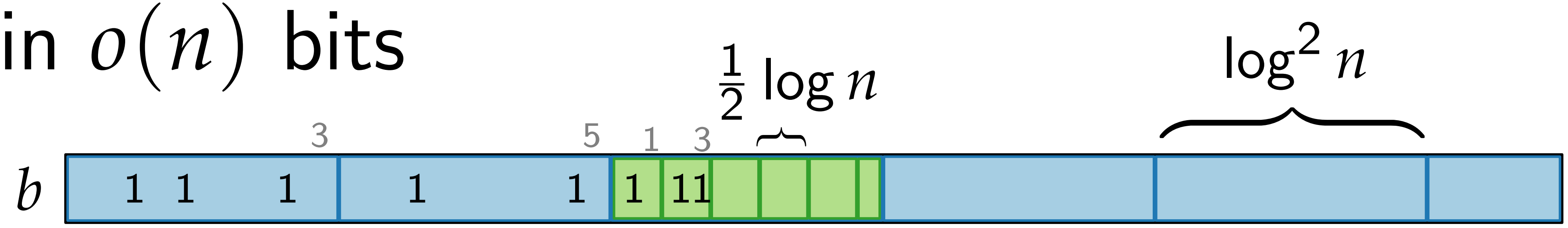

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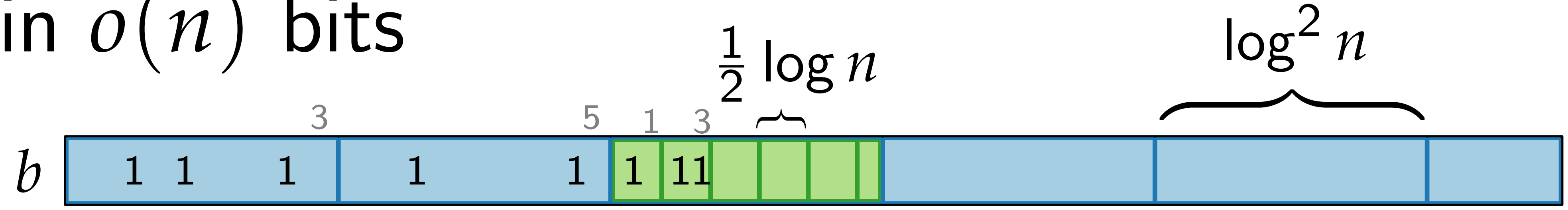
Rank in $o(n)$ bits



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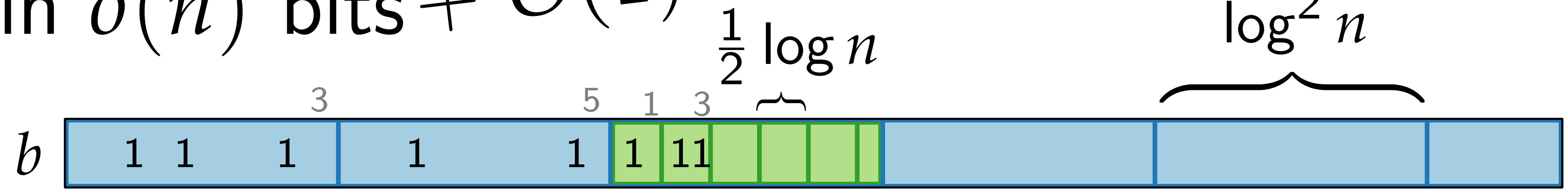
Rank in $o(n)$ bits + $O(1)$ time



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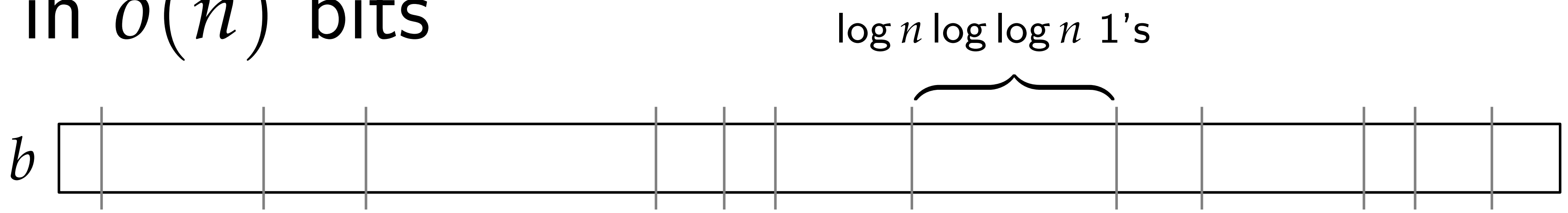
Select in $o(n)$ bits

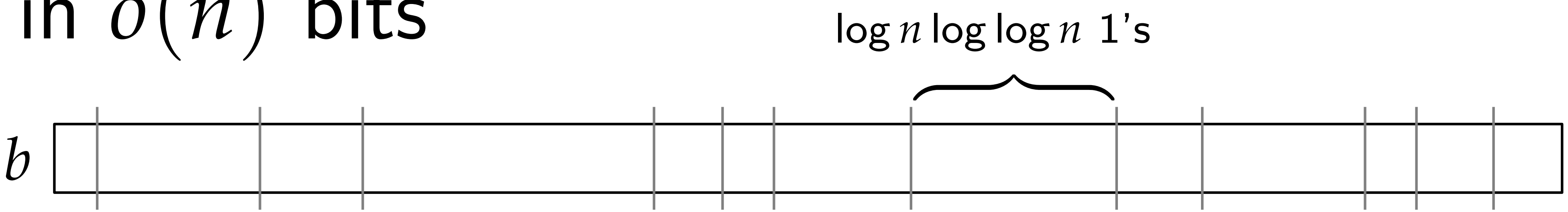
b

Select in $o(n)$ bits



1. Store indices of every $(\log n \log\log n)$th 1 bit in array

Select in $o(n)$ bits



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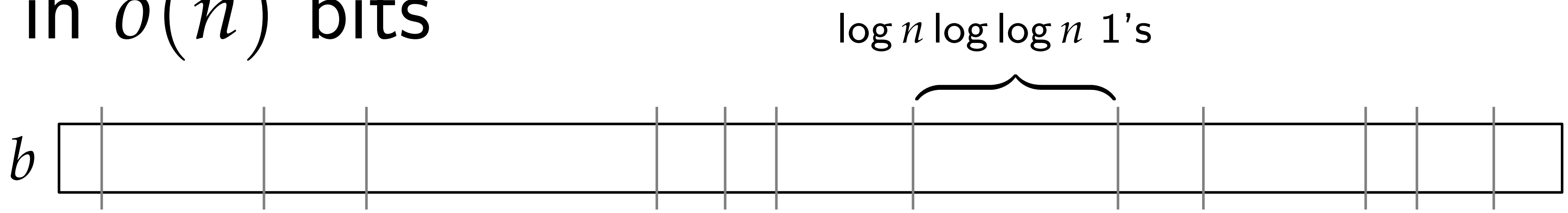

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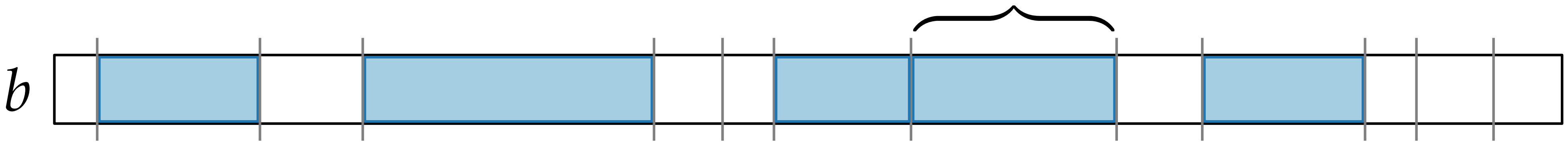

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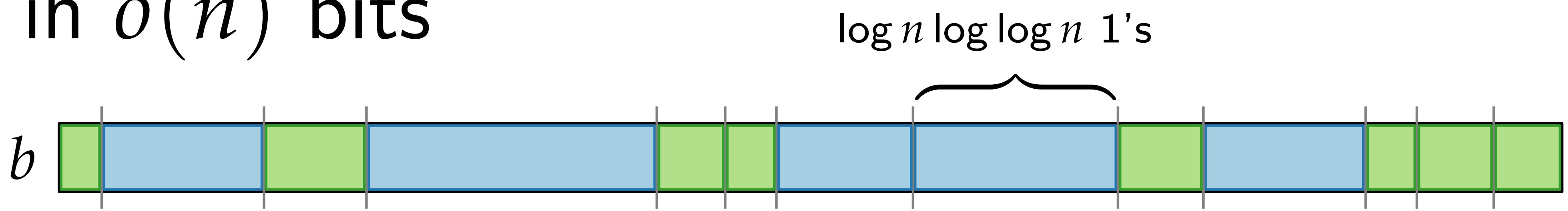
2. Within group of $(\log n \log\log n)$ 1 bits of length r bits:

   if $r \geq (\log n \log\log n)^2$
   then store indices of 1 bits in group in array
$$\Rightarrow O(\underbrace{\frac{n}{(\log n \log\log n)^2}}_{\#\text{ groups}} \underbrace{(\log n \log\log n)}_{\#\text{ 1 bits}} \underbrace{\log n}_{\text{index}}) \subseteq O(\frac{n}{\log\log n})$$

Select in $o(n)$ bits



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$$\Rightarrow O\left(\frac{n}{\log n \log\log n} \log n\right) = O\left(\frac{n}{\log\log n}\right) \subseteq o(n) \text{ bits}$$

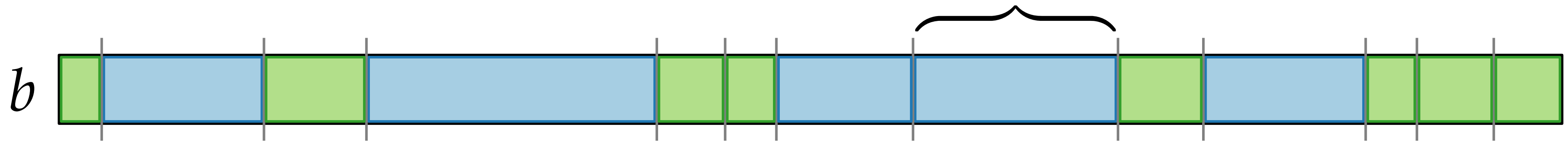
2. Within group of $(\log n \log\log n)$ 1 bits of length r bits:

   if $r \geq (\log n \log\log n)^2$
   then store indices of 1 bits in group in array

$$\Rightarrow O\left(\frac{n}{(\log n \log\log n)^2} (\log n \log\log n) \log n\right) \subseteq O\left(\frac{n}{\log\log n}\right)$$

   else problem is reduced to bitstrings of length $r < (\log n \log\log n)^2$

Select in $o(n)$ bits



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3. Repeat 1. and 2. on reduced bitstrings

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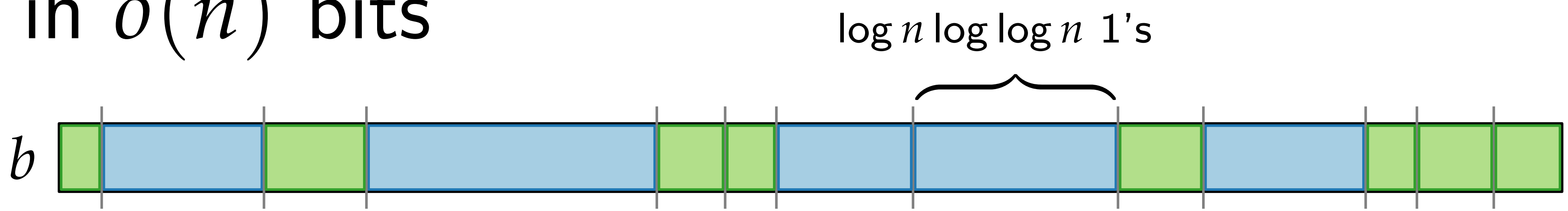

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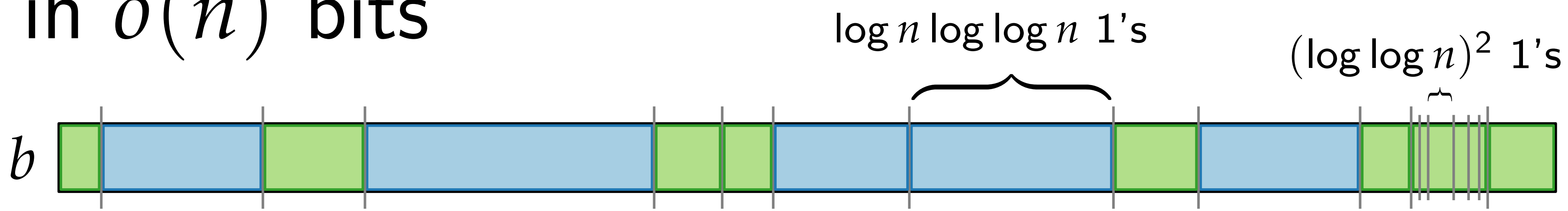
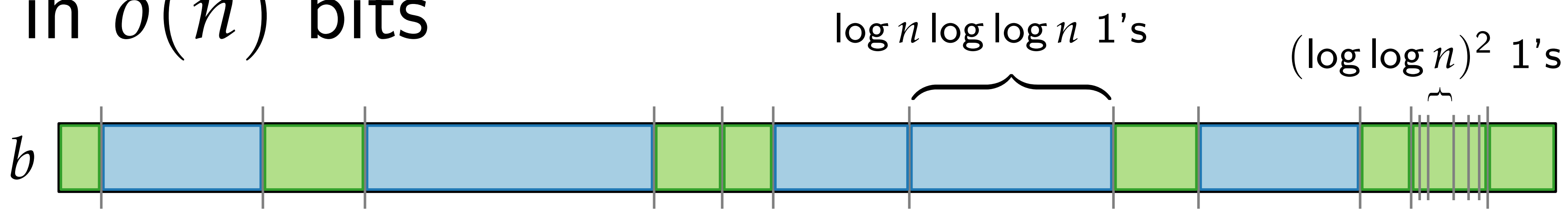


3. Repeat 1. and 2. on reduced bitstrings ($r < (\log n \log\log n)^2$):

   1' Store relative indices of every $(\log\log n)^2$th 1 bit in array

Select in $o(n)$ bits



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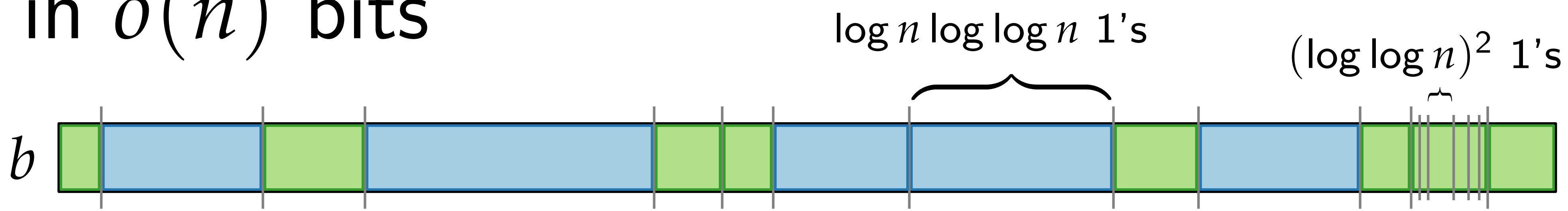
\# subgroups rel. index

Select in $o(n)$ bits



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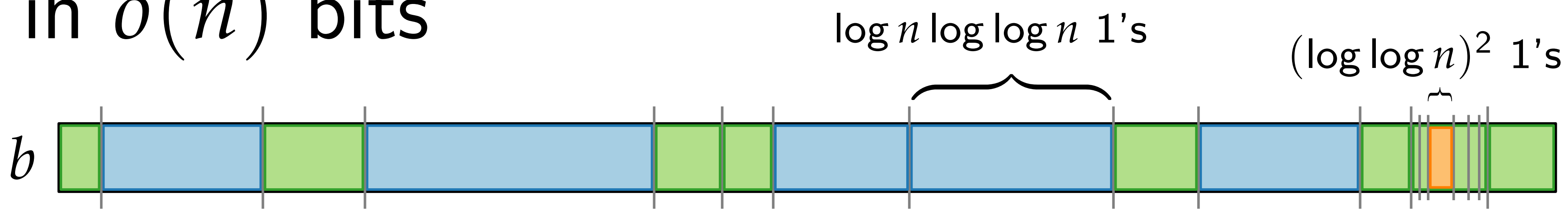
2' Within group of $(\log\log n)^2$th 1 bits of length r' bits:

Select in $o(n)$ bits



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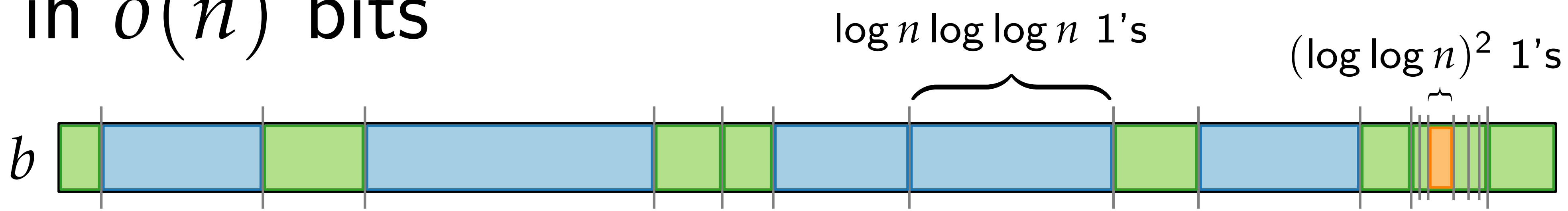
if $r' \geq (\log\log n)^4$

then store relative indices of 1 bits in subgroup in array

Select in $o(n)$ bits



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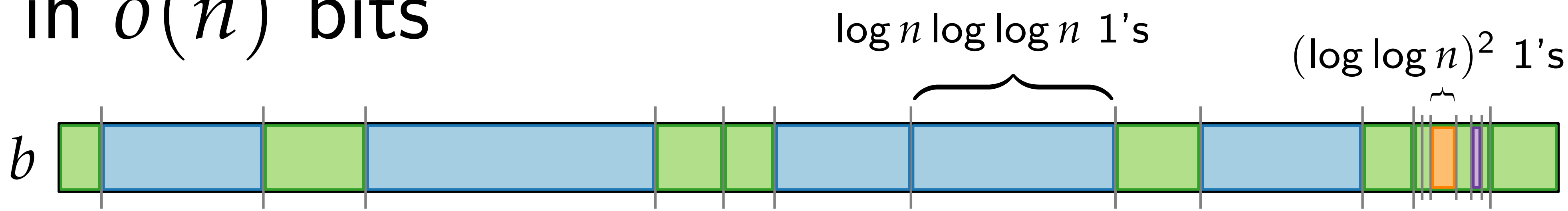
Select in $o(n)$ bits



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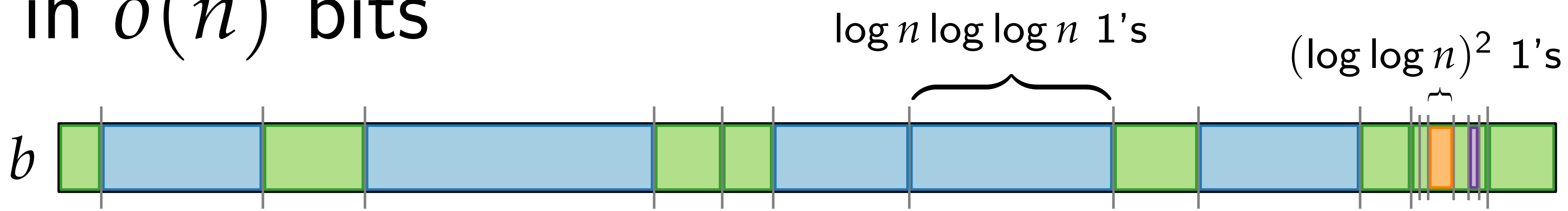

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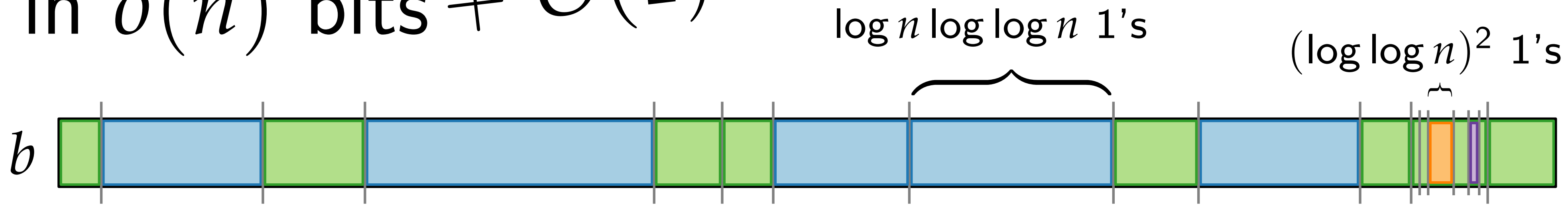
else problem is reduced to bitstrings of length $r' < (\log\log n)^4$

4. Use lookup table for bitstrings of length $r' \leq (\log\log n)^4 \leq \frac{1}{2}\log n$

$$\Rightarrow O(\sqrt{n}\log n \log\log n) = o(n) \text{ bits}$$

bitstring query j answer

Select in $o(n)$ bits + $O(1)$ time



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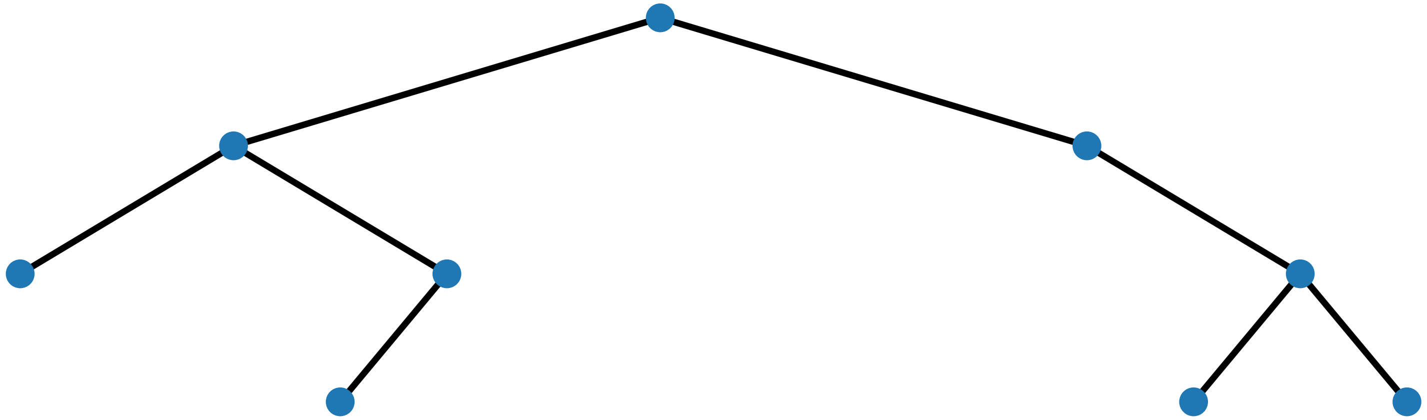
bitstring query j answer

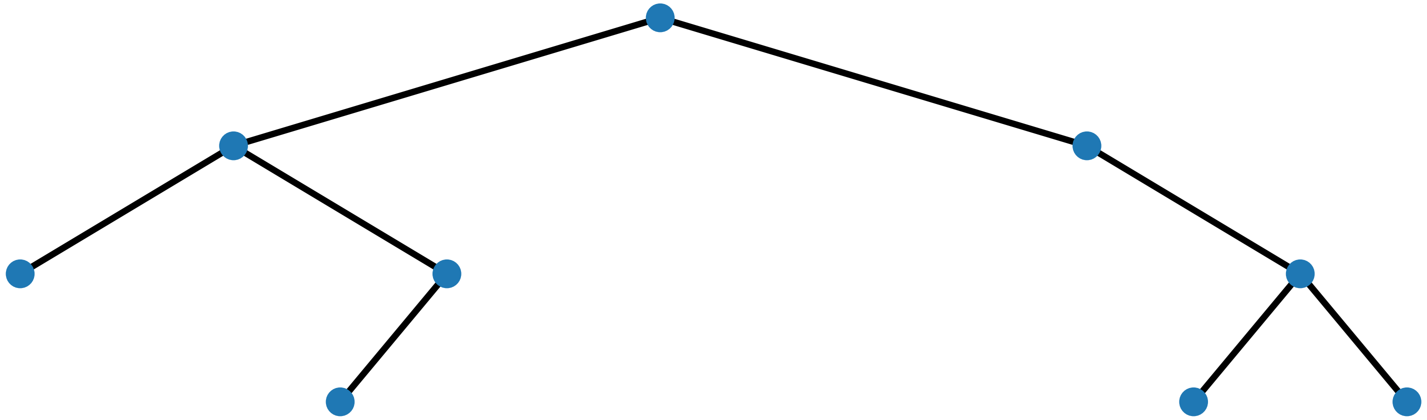
Succinct representation of binary trees



Number of binary trees on n vertices: $C_n = \frac{1}{n+1}\binom{2n}{n}$

$$\log C_n = 2n + o(n) \text{ (by Stirling's approximation)}$$

Succinct representation of binary trees



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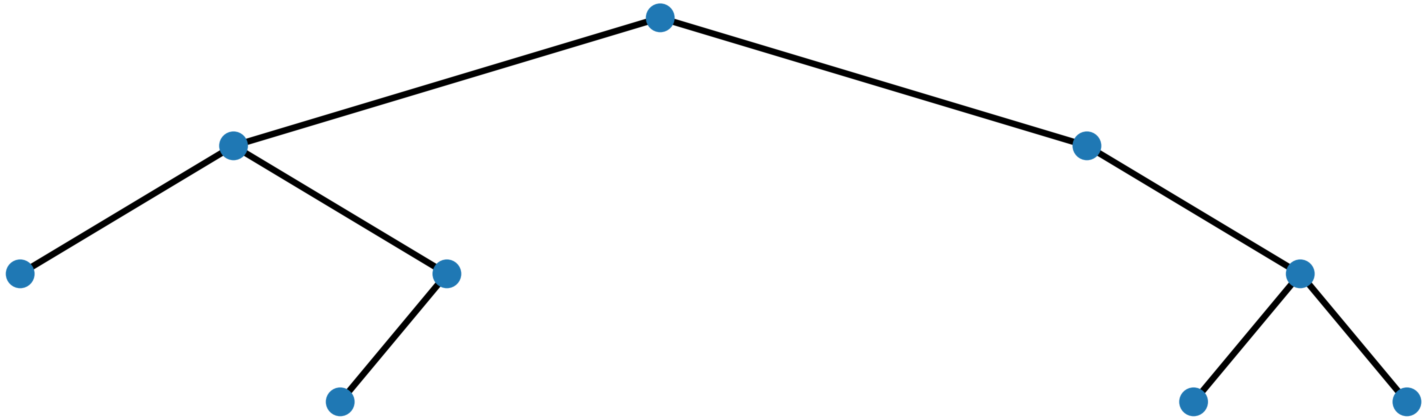

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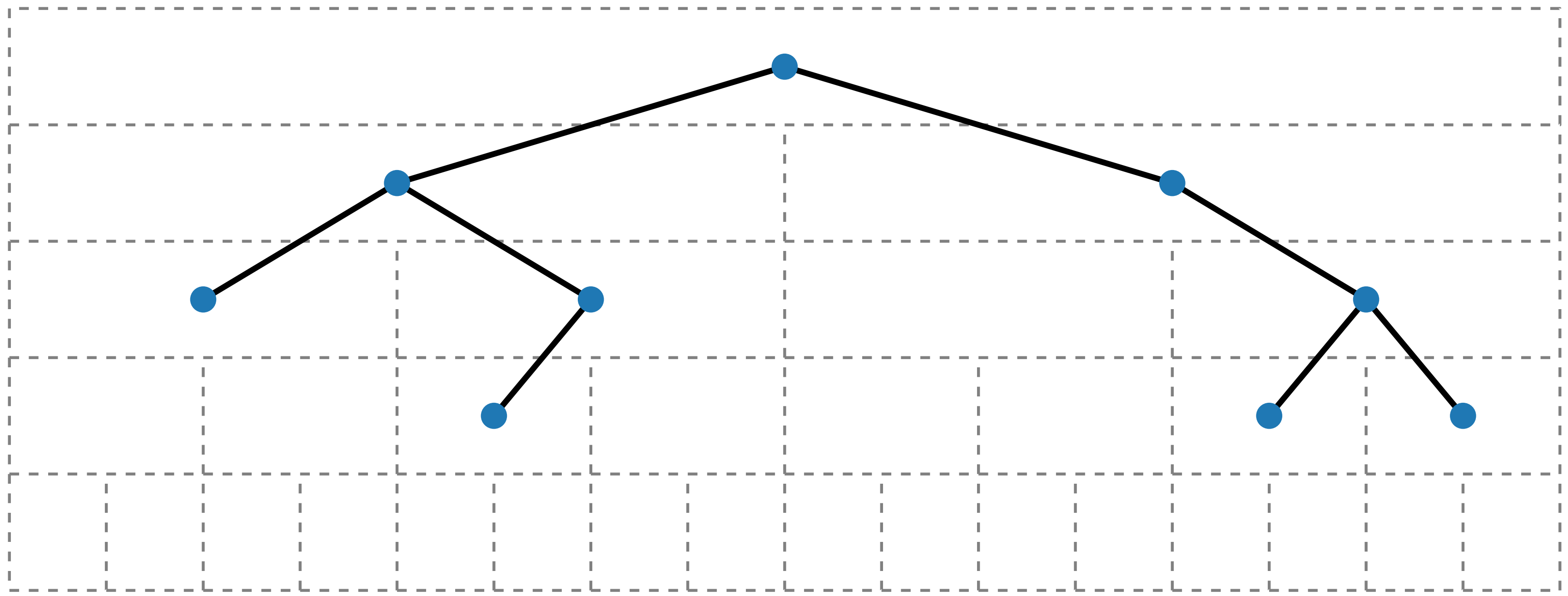
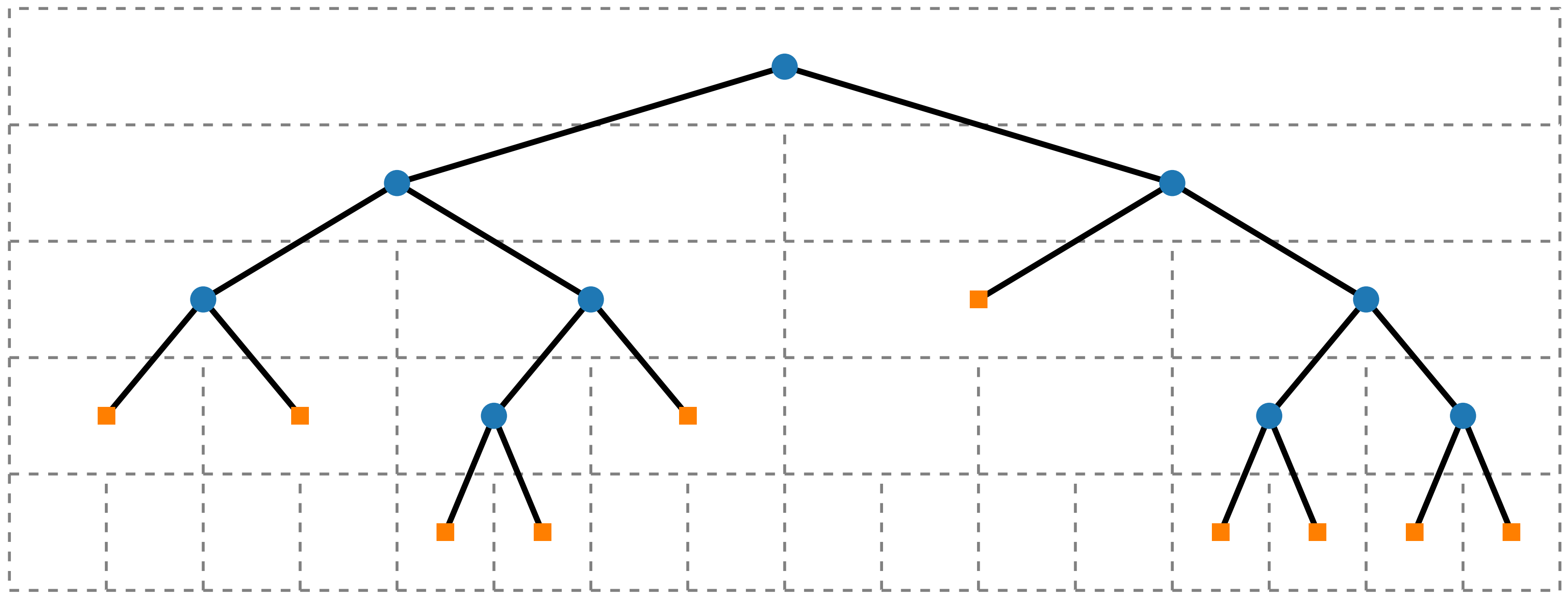
Difficulty is when binary tree is not full.

Succinct representation of binary trees



Idea.

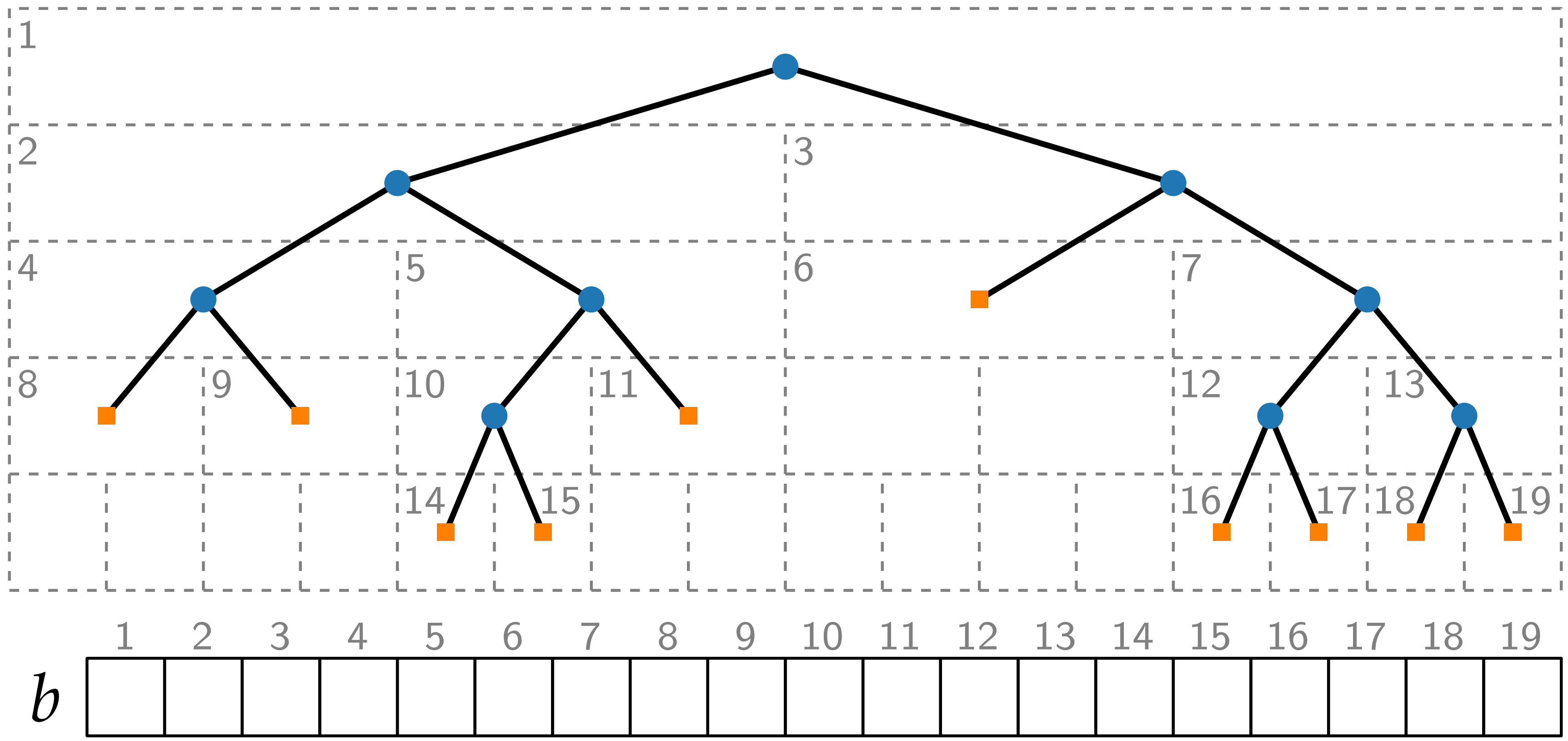
Succinct representation of binary trees



Idea.

- Add external nodes

Succinct representation of binary trees



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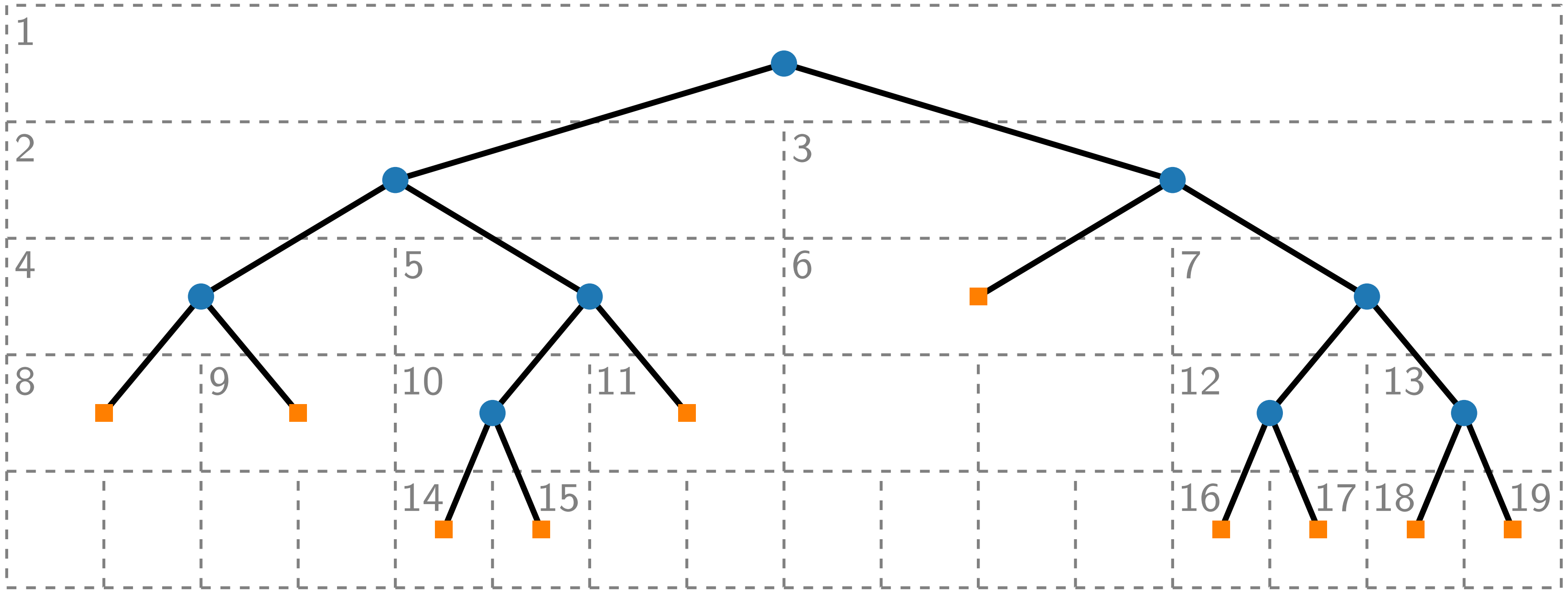
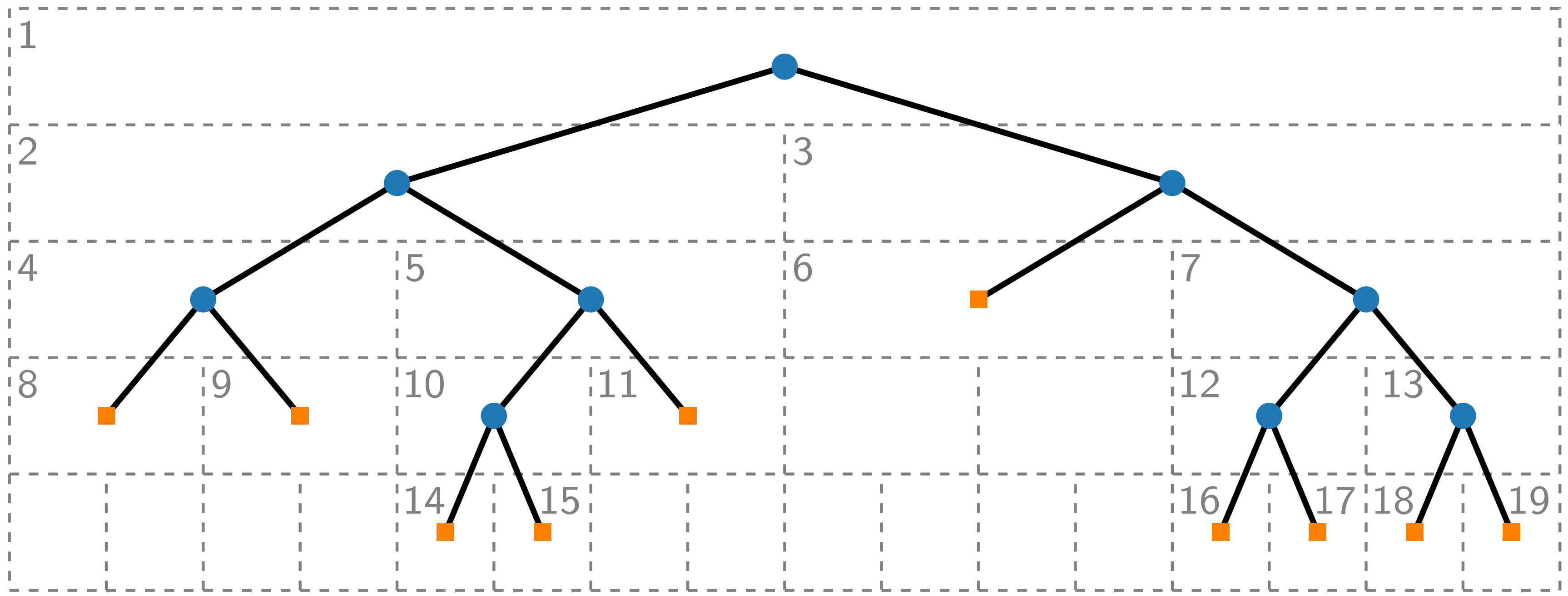


	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
b	1	1	1	1	1	0	1	0	0	1	0	1	1	0	0	0	0	0	0

Idea.

- Add external nodes
- Read internal nodes as 1
- Read external nodes as 0

Succinct representation of binary trees



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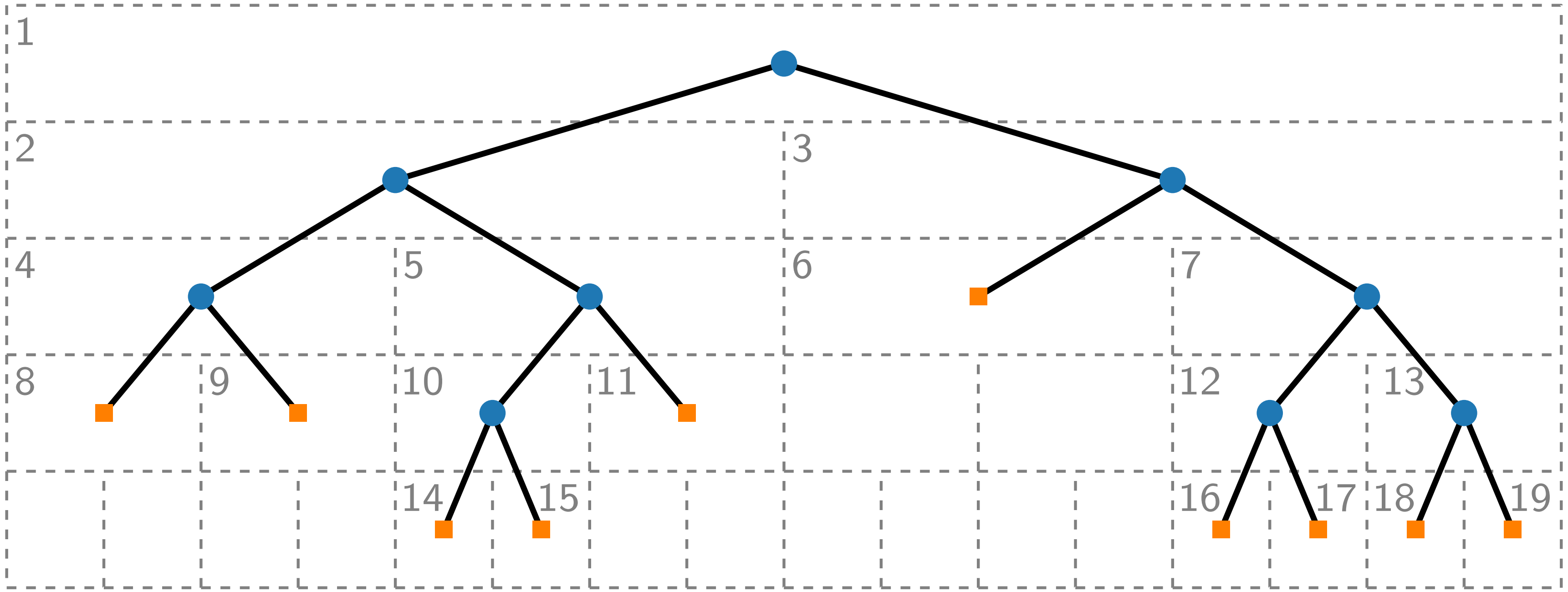
Operations.

- $\texttt{parent}(i) = \ ?$
- $\texttt{leftChild}(i) = \ ?$
- $\texttt{rightChild}(i) = ?$

Succinct representation of binary trees



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Idea.

- Add external nodes
- Read internal nodes as 1
- Read external nodes as 0
- Use `rank` and `select`

Operations.

- $\texttt{parent}(i) =$?
- $\texttt{leftChild}(i) =$?
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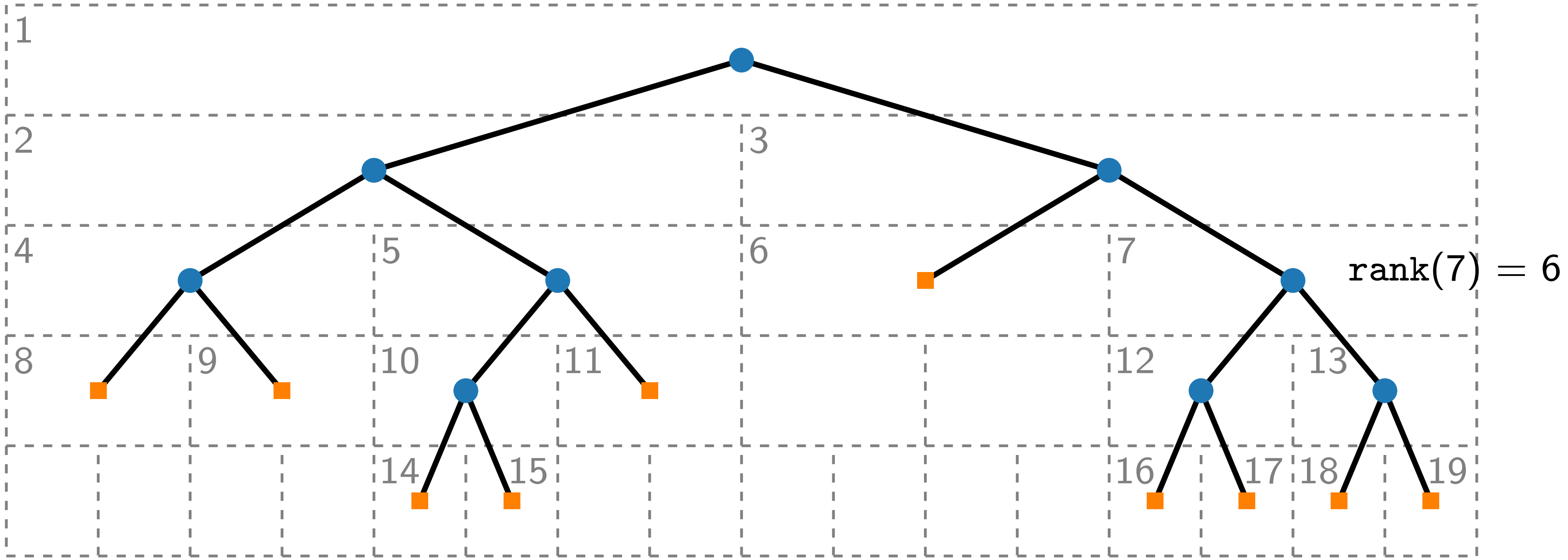
Succinct representation of binary trees



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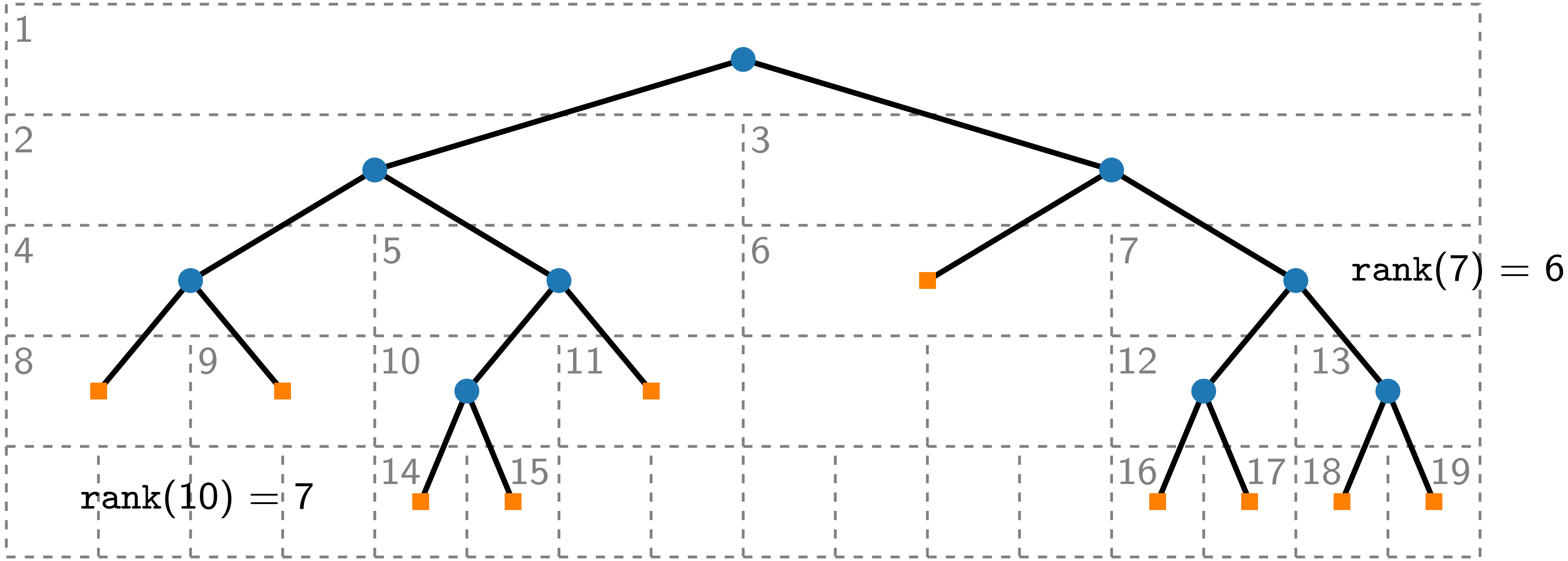

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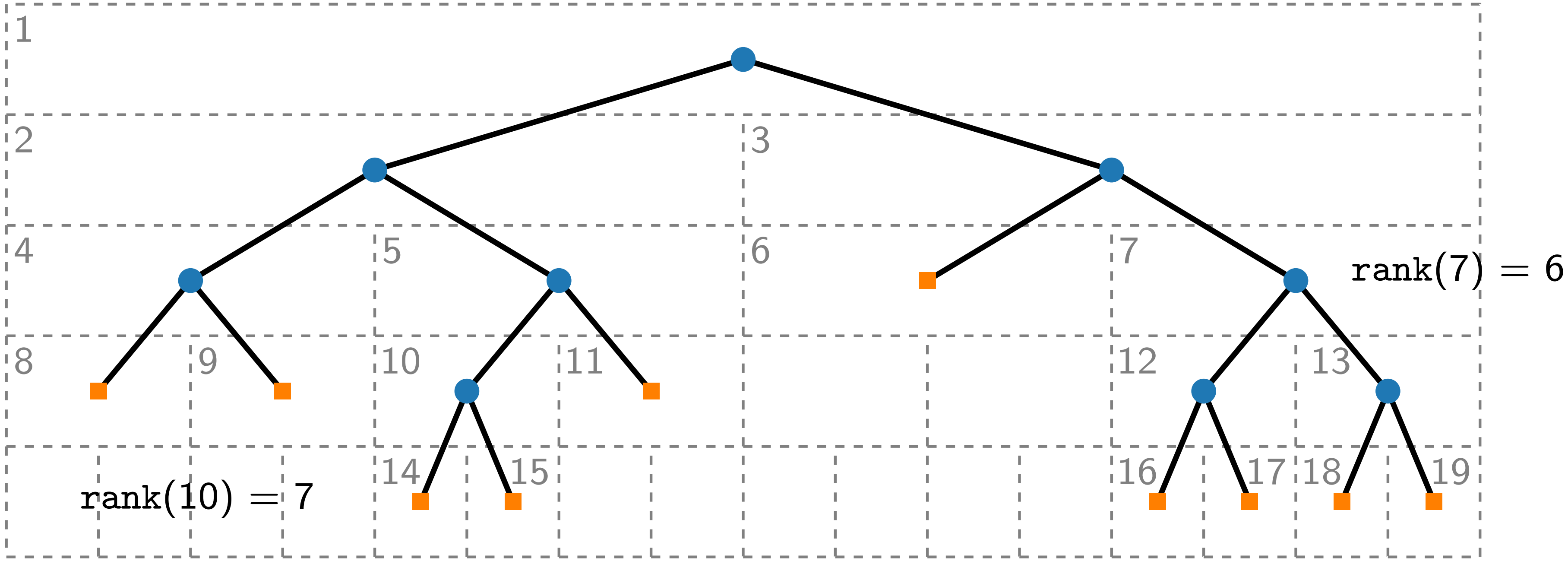

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Idea.

- Add external nodes
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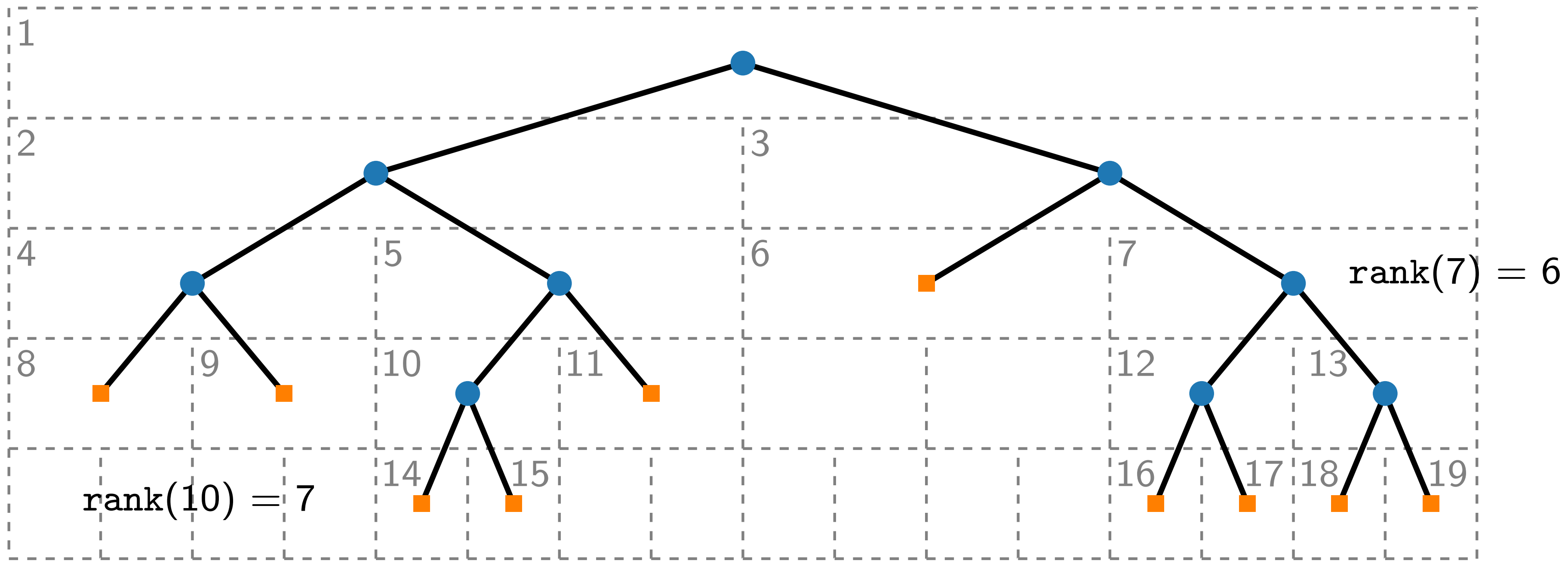
- Use rank and select

Operations.

- $\texttt{parent}(i) = \quad ?$
- $\texttt{leftChild}(i) = 2\,\texttt{rank}(i)$
- $\texttt{rightChild}(i) = 2\,\texttt{rank}(i) + 1$

Succinct representation of binary trees



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Idea.

- Add external nodes
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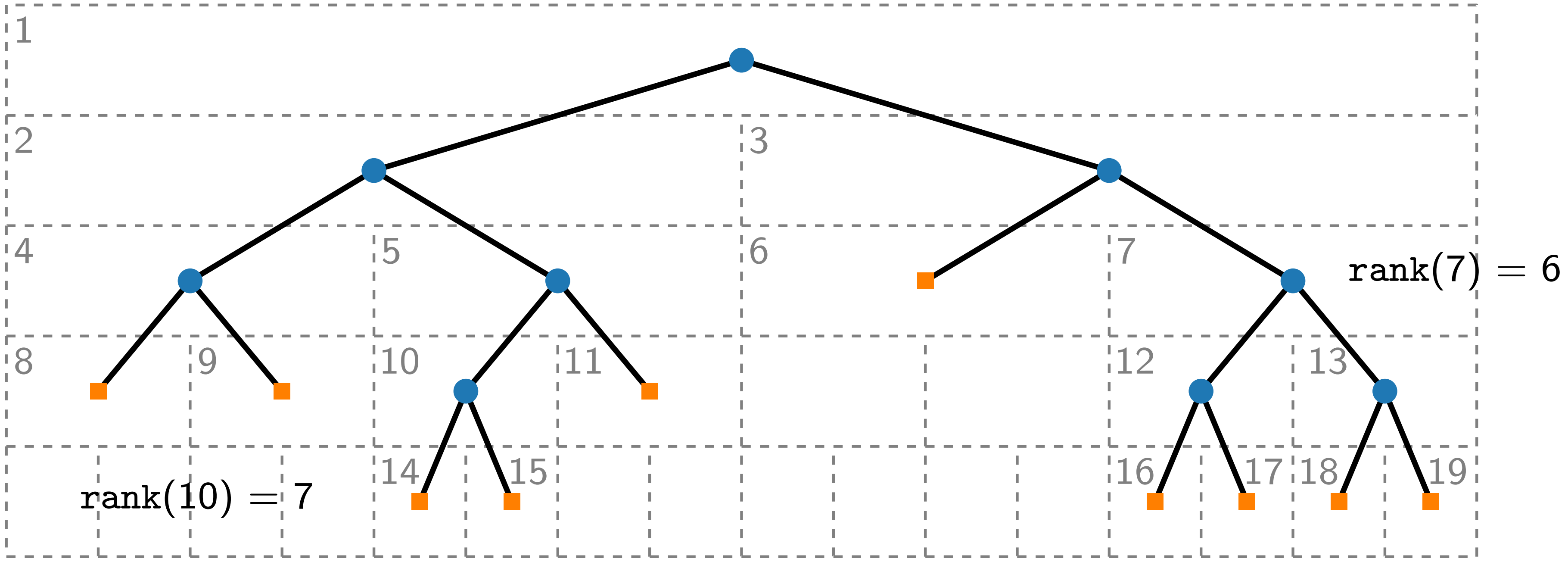
- Use rank and select

Operations.

- $\texttt{parent}(i) = \texttt{select}(\lfloor \frac{i}{2} \rfloor)$
- $\texttt{leftChild}(i) = 2\,\texttt{rank}(i)$
- $\texttt{rightChild}(i) = 2\,\texttt{rank}(i) + 1$

Succinct representation of binary trees



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Idea.

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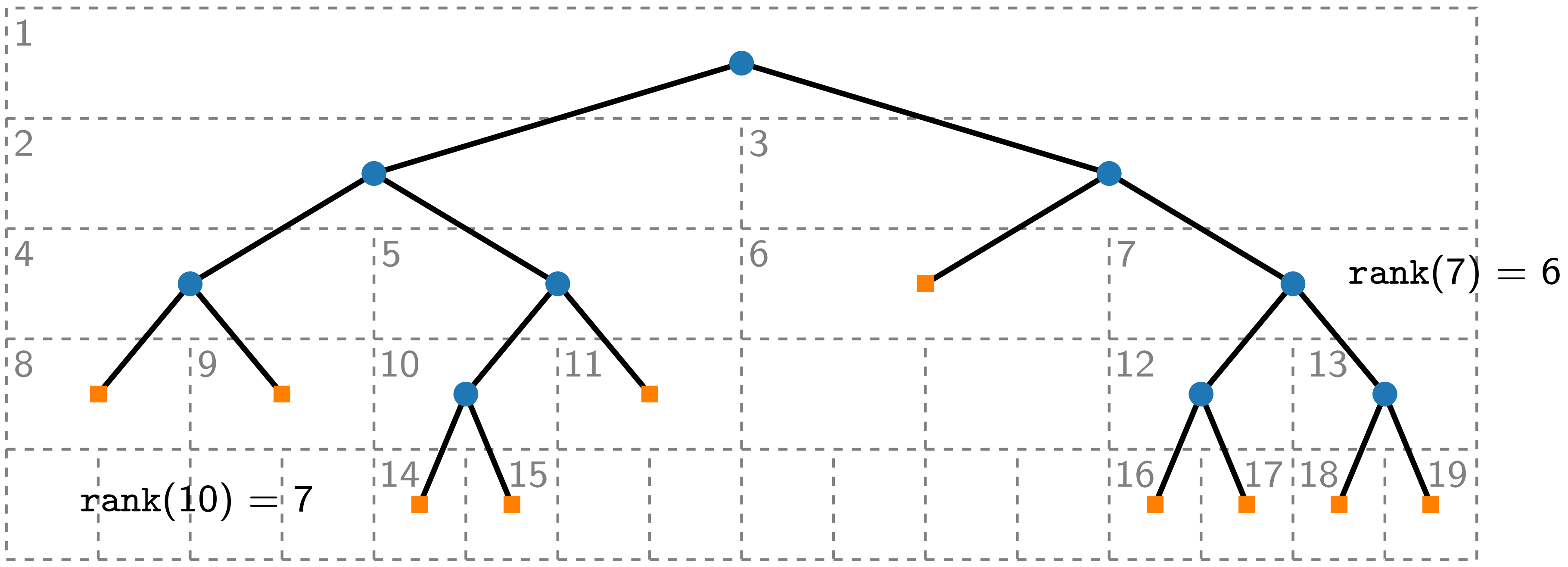
Proof is exercise.

Succinct representation of binary trees



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Idea.

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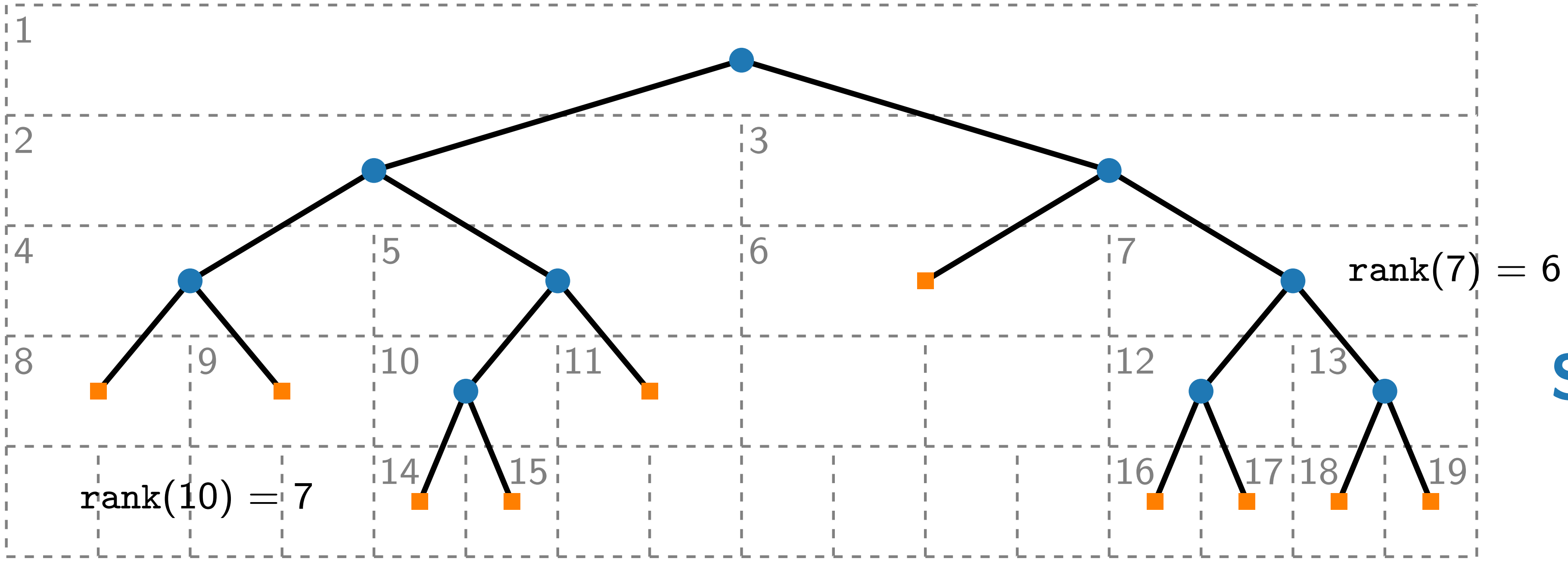
- $\texttt{rank}(i)$ is index for array storing actual values

Proof is exercise.

Succinct representation of binary trees



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b	1	1	1	1	1	0	1	0	0	1	0	1	1	0	0	0	0	0	0

Size.
- 2n +1 bits for b
- $o(n)$ for `rank` and `select`

Idea.
- Add external nodes
- Read internal nodes as 1
- Read external nodes as 0
- Use `rank` and `select`

Operations.
- $\texttt{parent}(i) = \texttt{select}(\lfloor \frac{i}{2} \rfloor)$
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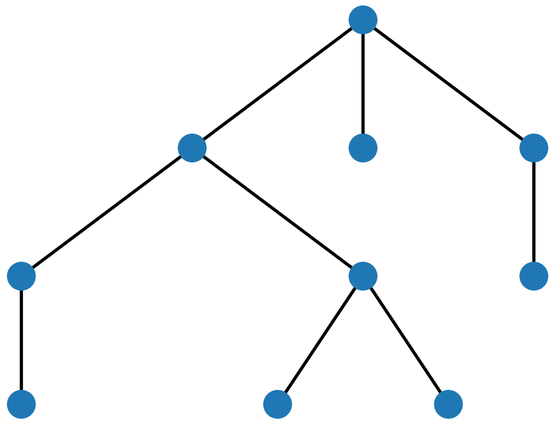
Proof is exercise.

Succinct representation of trees - LOUDS

LOUDS = Level Order Unary Degree Sequence



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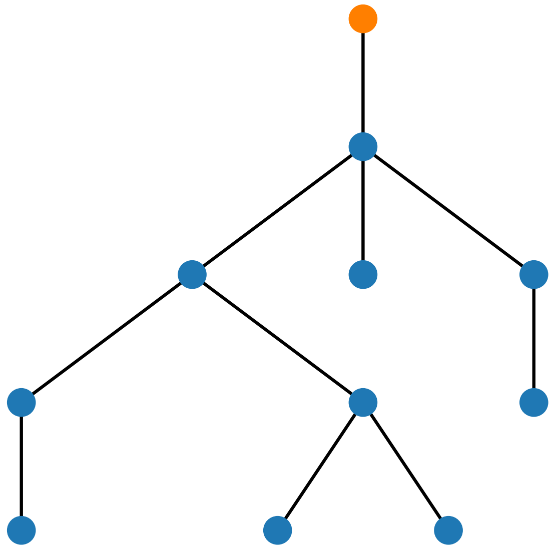

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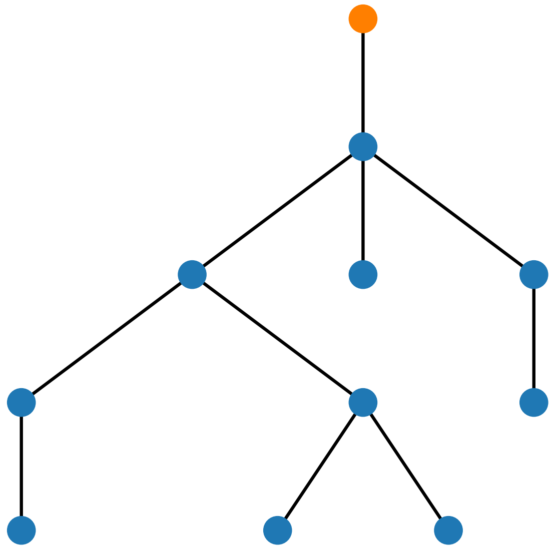

- unary decoding of outdegree

Succinct representation of trees - LOUDS

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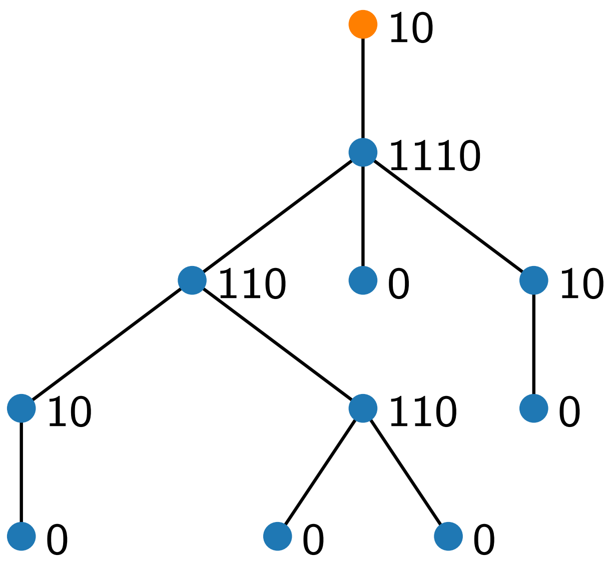

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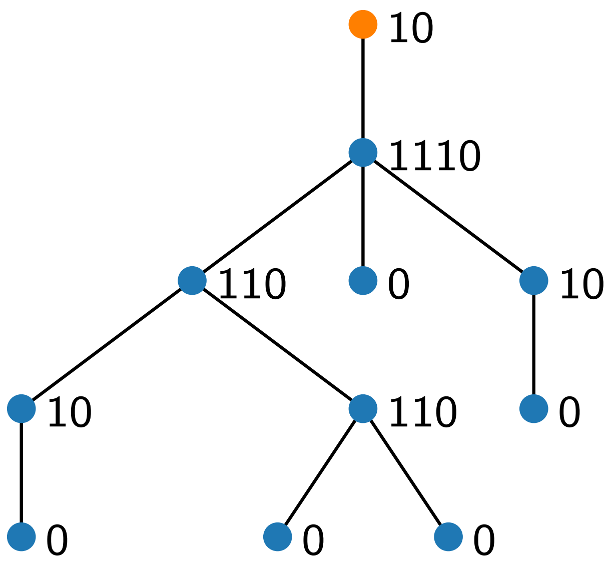

- unary decoding of outdegree
- gives LOUDS sequence

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	0	1	1	1	0	1	1	0	0	1	0	1	0	1	1	0	0	0	0	0

Succinct representation of trees - LOUDS

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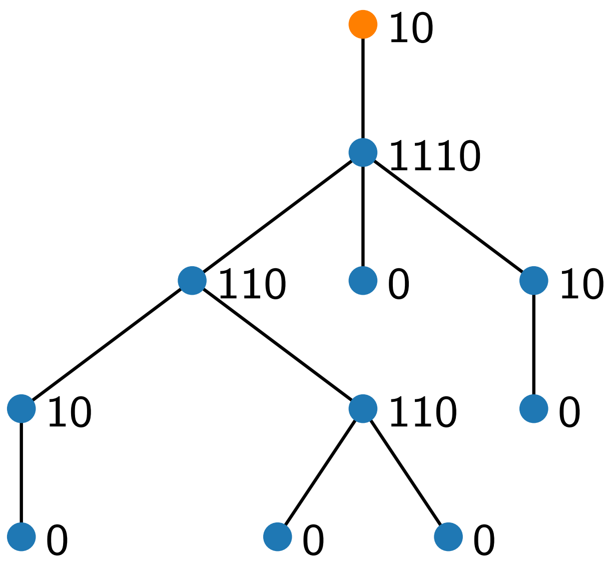

- unary decoding of outdegree
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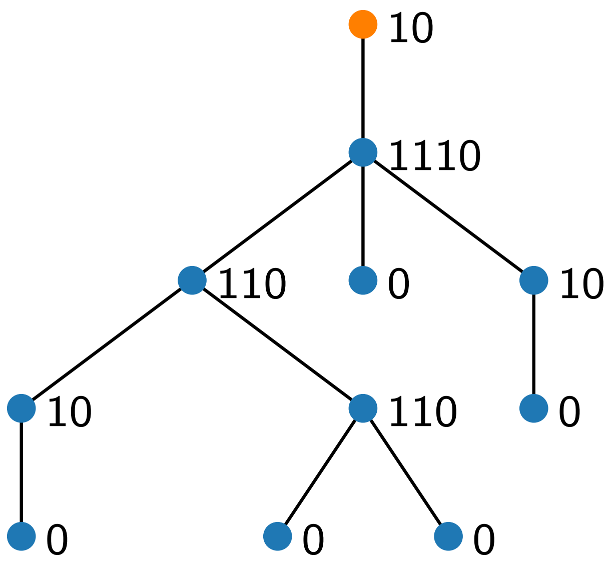
1	0	1	1	1	0	1	1	0	0	1	0	1	0	1	1	0	0	0	0	0

Size.

- each vertex (except root) is represented twice, namely with a 1 and with a 0
- $o(n)$ bits for `rank` and `select`

Succinct representation of trees - LOUDS

LOUDS = Level Order Unary Degree Sequence



- unary decoding of outdegree
- gives LOUDS sequence

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	0	1	1	1	0	1	1	0	0	1	0	1	0	1	1	0	0	0	0	0

Size.

- each vertex (except root) is represented twice, namely with a 1 and with a 0
- $o(n)$ bits for `rank` and `select`

$\Rightarrow 2n + o(n)$ bits

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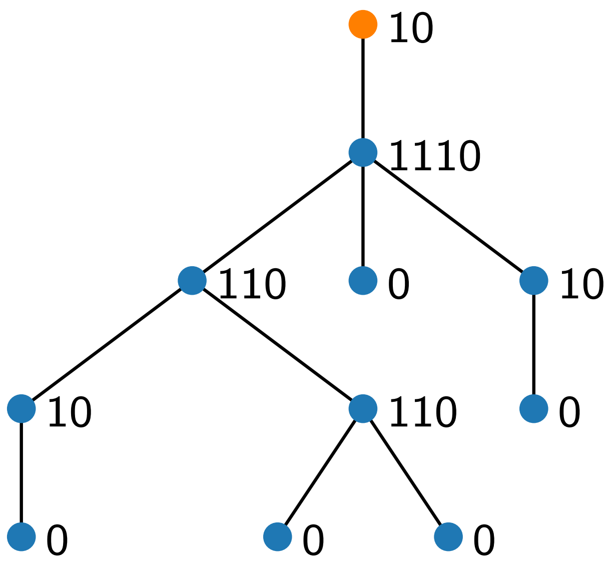

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Operations.

- Let i be index of 1 in louds sequence.
- $\texttt{rank}(i)$ is index for array storing vertex objects/values.

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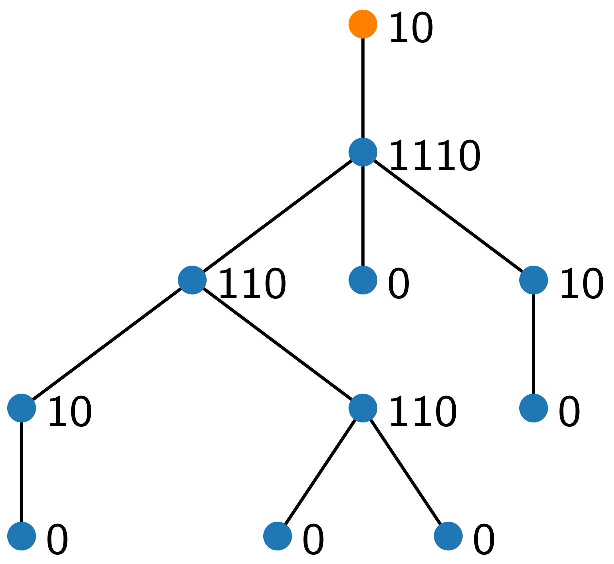

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- $\texttt{firstChild}(i) = \texttt{select}_0(\texttt{rank}_1(i)) + 1$

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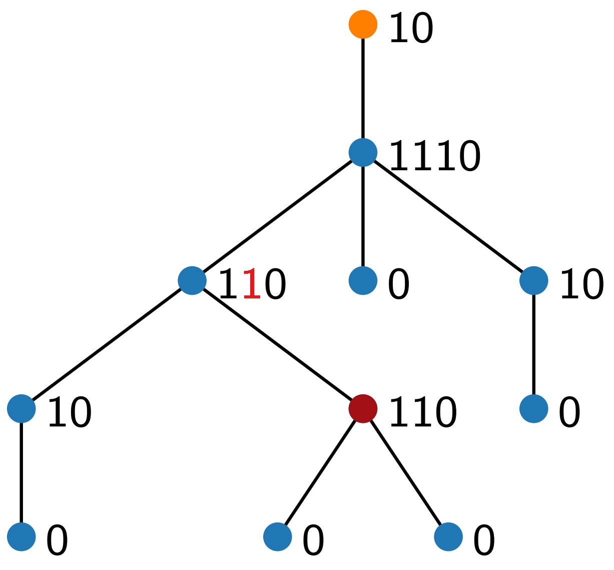

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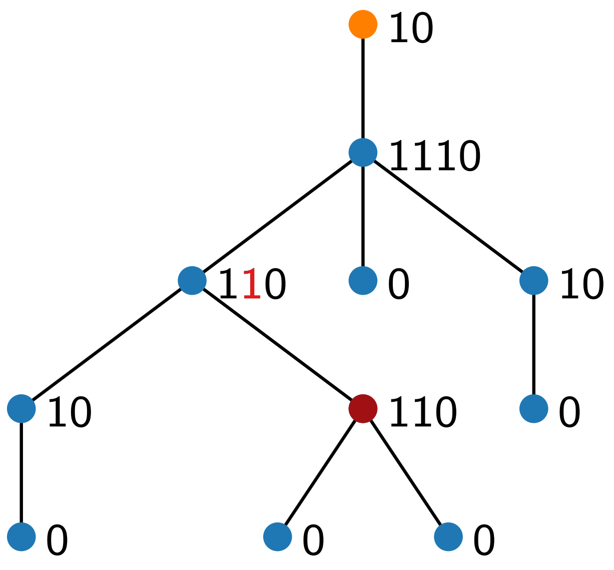

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$$\texttt{firstChild}(8) = \texttt{select}_0(\texttt{rank}_1(8)) + 1$$
$$= \texttt{select}_0(6) + 1$$

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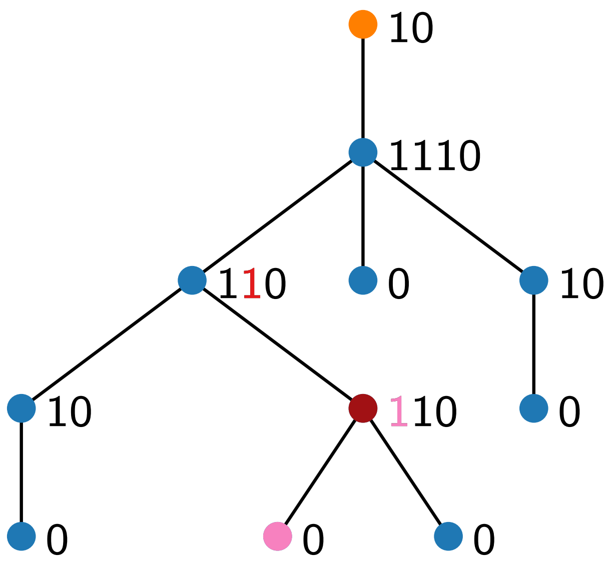

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$$= \texttt{select}_0(6) + 1 = 14 + 1 = 15$$

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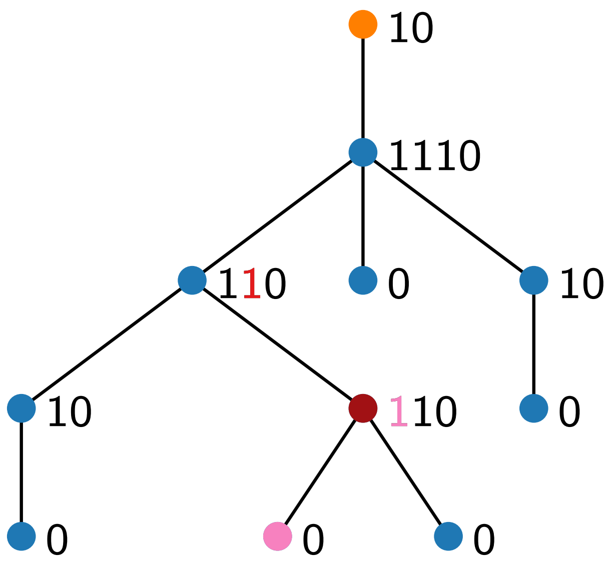

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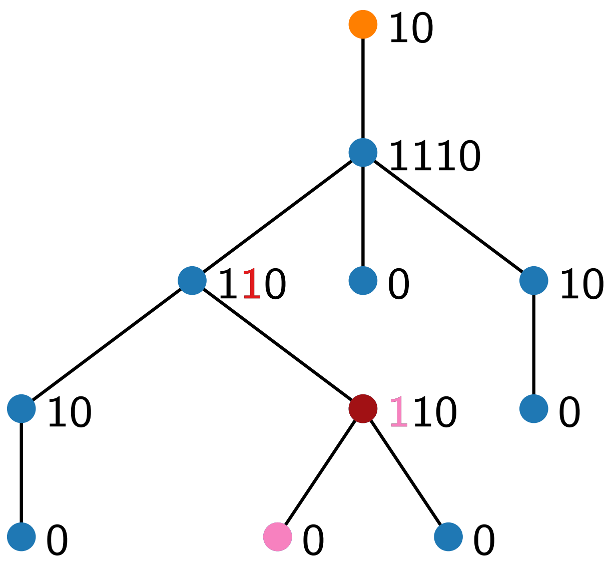

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Exercise: $\texttt{child}(i, j)$
with validity check

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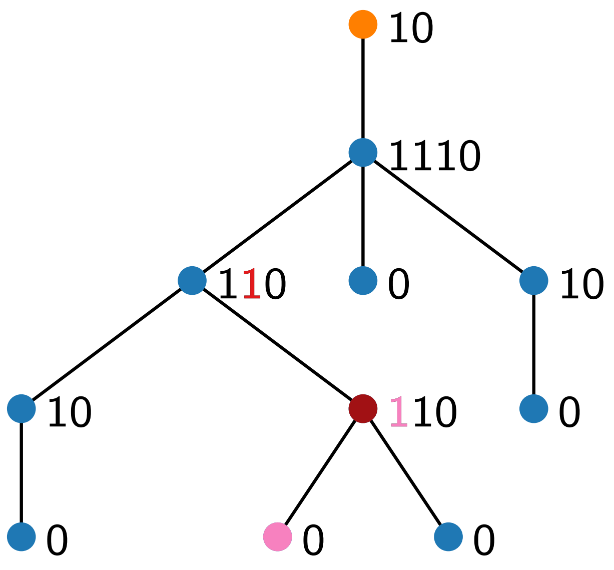

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- $\texttt{parent}(i) = \texttt{select}_1(\texttt{rank}_0(i))$

- $\texttt{nextSibling}(i) = i + 1$

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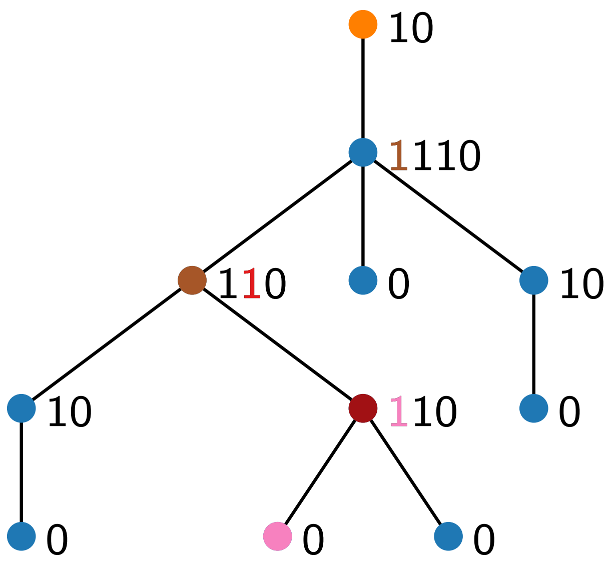

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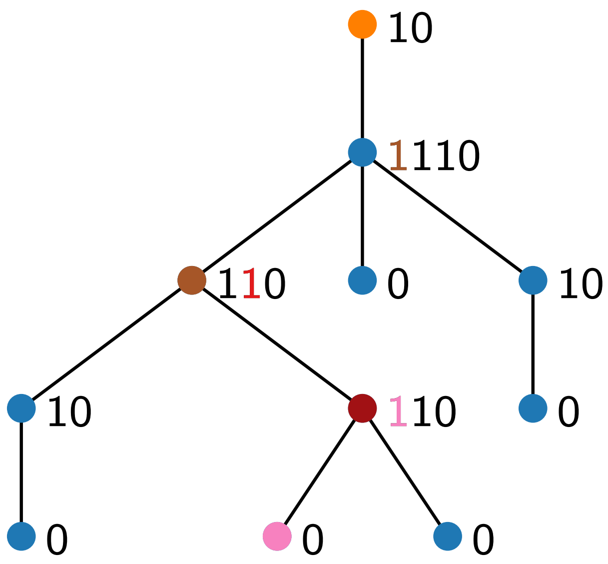

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- $\texttt{nextSibling}(i) = i + 1$

- $\texttt{parent}(i) = \texttt{select}_1(\texttt{rank}_0(i))$

  $\texttt{parent}(8) = \texttt{select}_1(\texttt{rank}_0(8))$
  $= \texttt{select}_1(2)$

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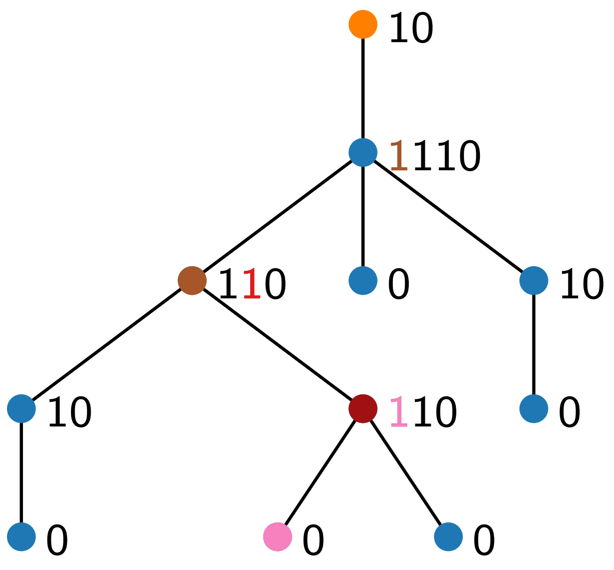

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  $= \texttt{select}_0(6) + 1 = 14 + 1 = 15$

- $\texttt{parent}(i) = \texttt{select}_1(\texttt{rank}_0(i))$

  $\texttt{parent}(8) = \texttt{select}_1(\texttt{rank}_0(8))$
  $= \texttt{select}_1(2) = 3$

- $\texttt{nextSibling}(i) = i + 1$

Exercise: $\texttt{child}(i, j)$
with validity check

Discussion

- Succinct data structures are
  - space efficient
  - support fast operations

  but
  - are mostly static (dynamic at extra cost),
  - number of operations are limited,
  - complex $\rightarrow$ harder to implement

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- Rank and select form basis for many succinct representations

Literature

Main reference:

- Lecture 17 of Advanced Data Structures (MIT, Fall'17) by Erik Demaine
- [Jac '89] "Space efficient Static Trees and Graphs"

Recommendations:

- Lecture 18 of Demaine's course on compact & succinct arrays & trees