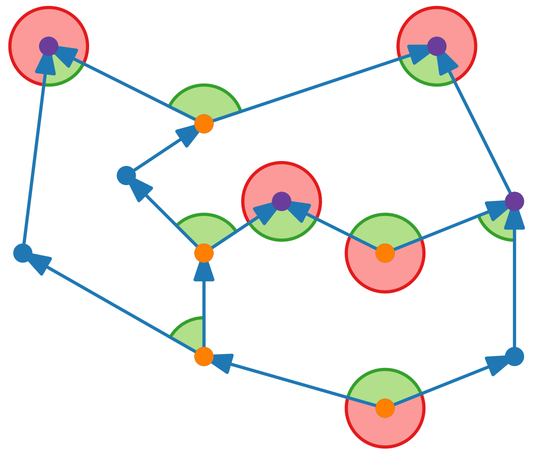
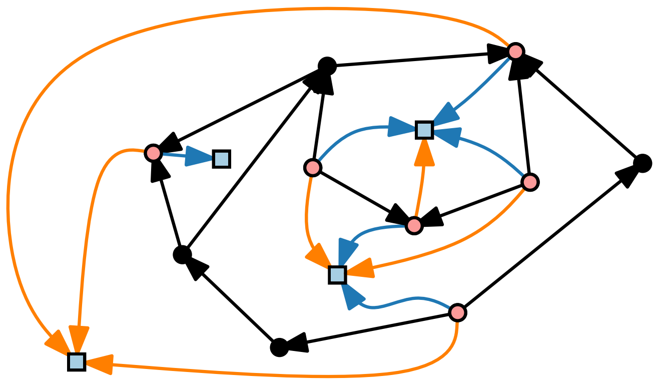
# Visualisation of graphs

## Upward planar drawings

### Flow methods

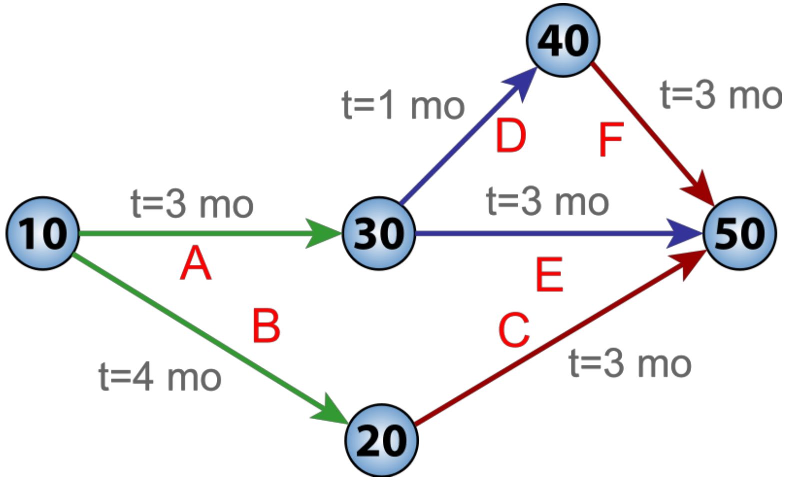
Jonathan Klawitter · Summer semester 2020



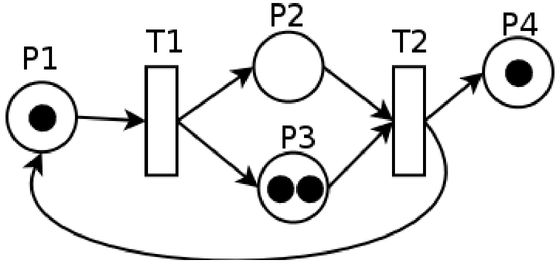


# Upward planar drawings – motivation

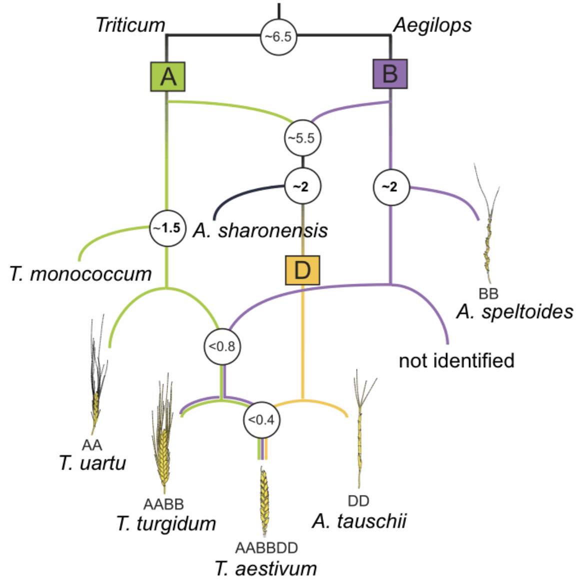
- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchie
  - . . .




PERT diagram



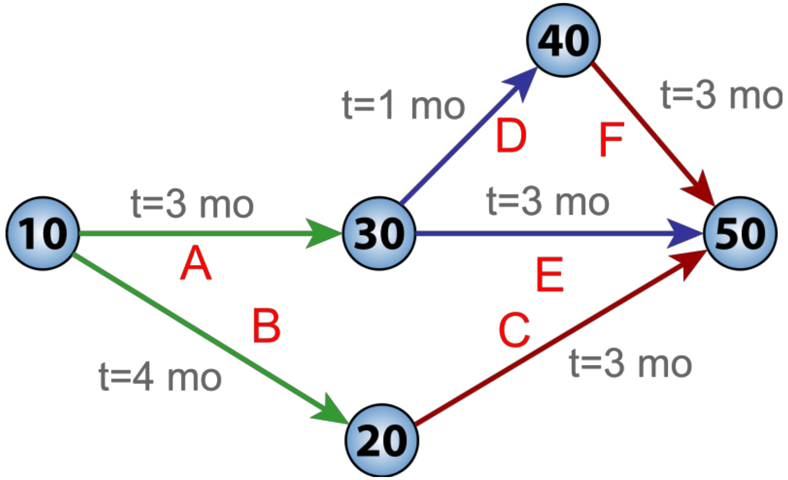

Petri net




Phylogenetic network

# Upward planar drawings – motivation

- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchie
  - ...
- Would be nice to have general direction preserved in drawing.

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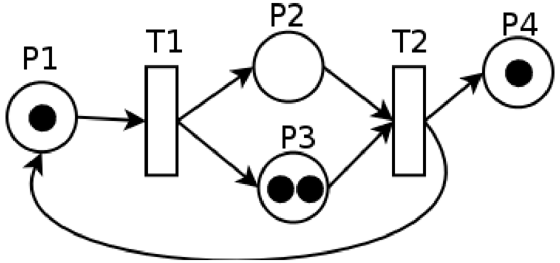


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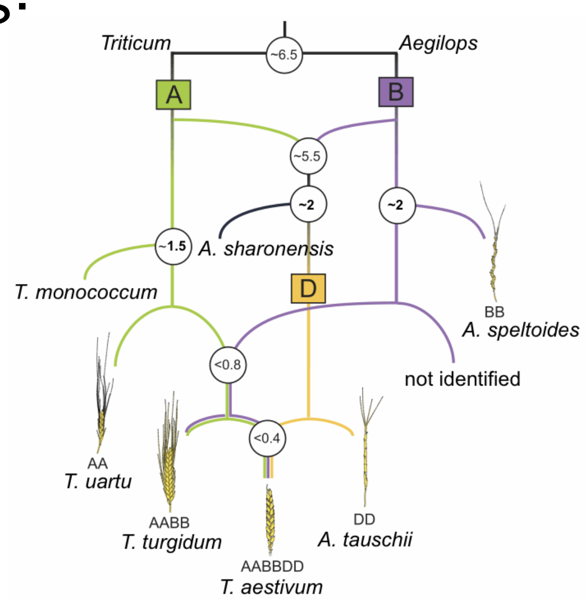
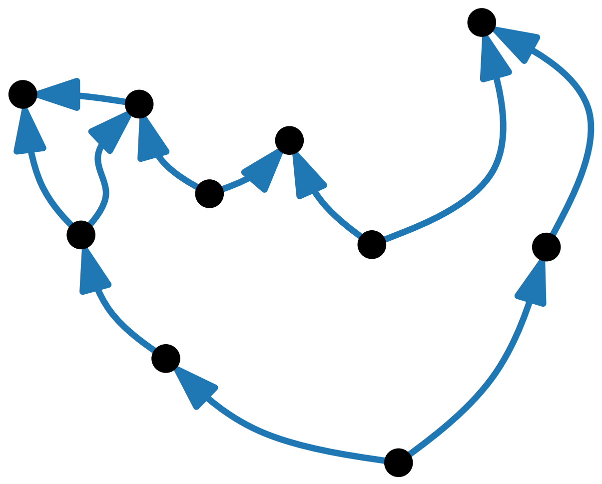



Phylogenetic network

# Upward planar drawings – definition

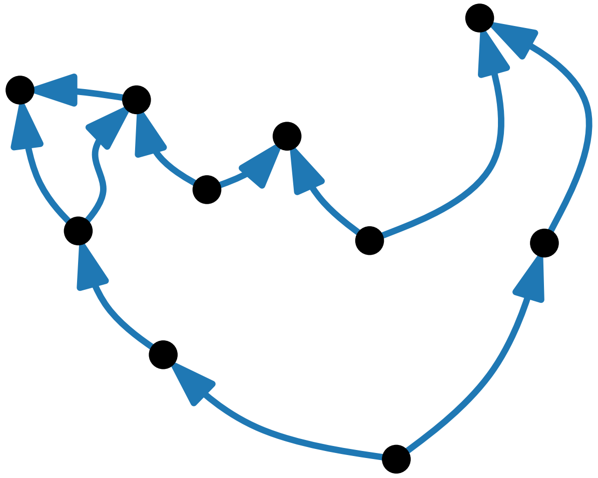
**Definition.**
A directed graph $G = (V, E)$ is **upward planar** when it admits a drawing $\Gamma$ (vertices = points, edges = simple curves) that is

- planar and
- where each edge is drawn as an upward, y-monotone curve.



# Upward planarity – necessary conditions

- For a digraph $G$ to be upward planar, it has to be:
  - planar



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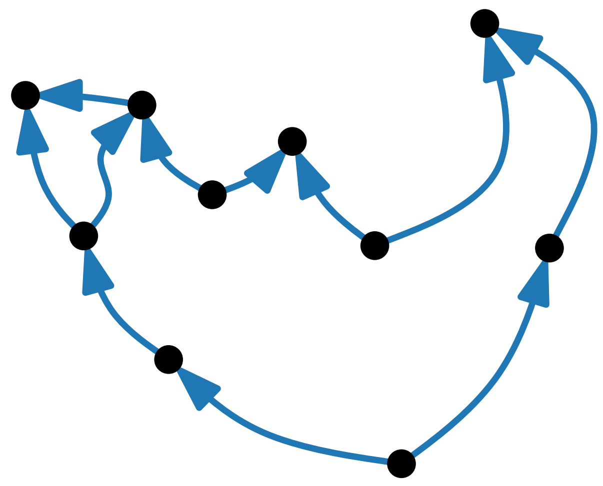

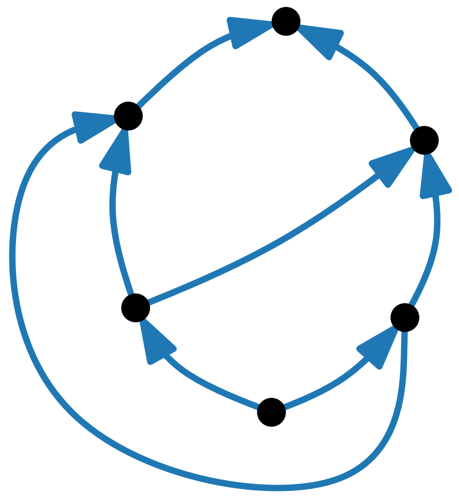

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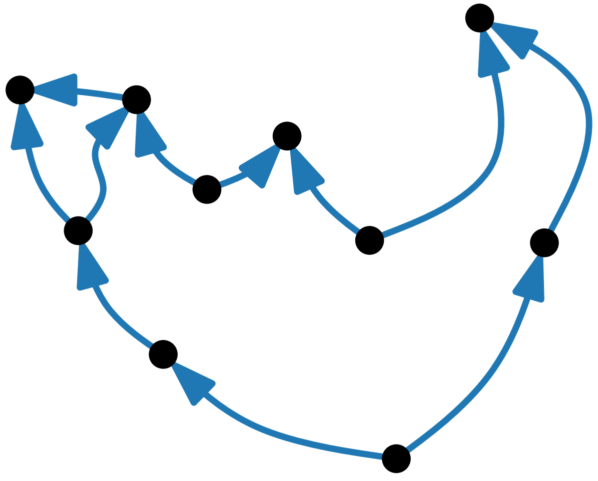
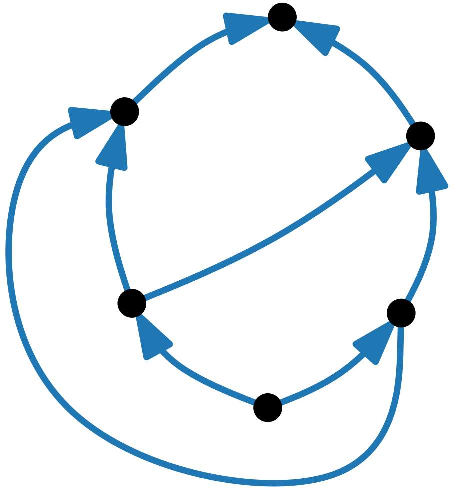
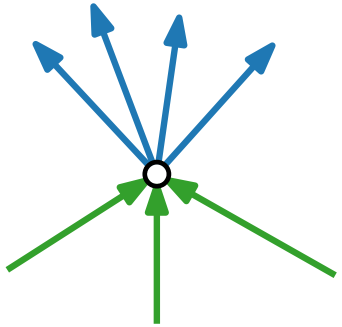




**bimodal** vertex



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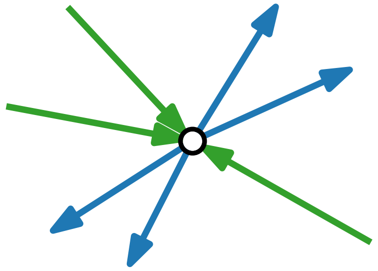

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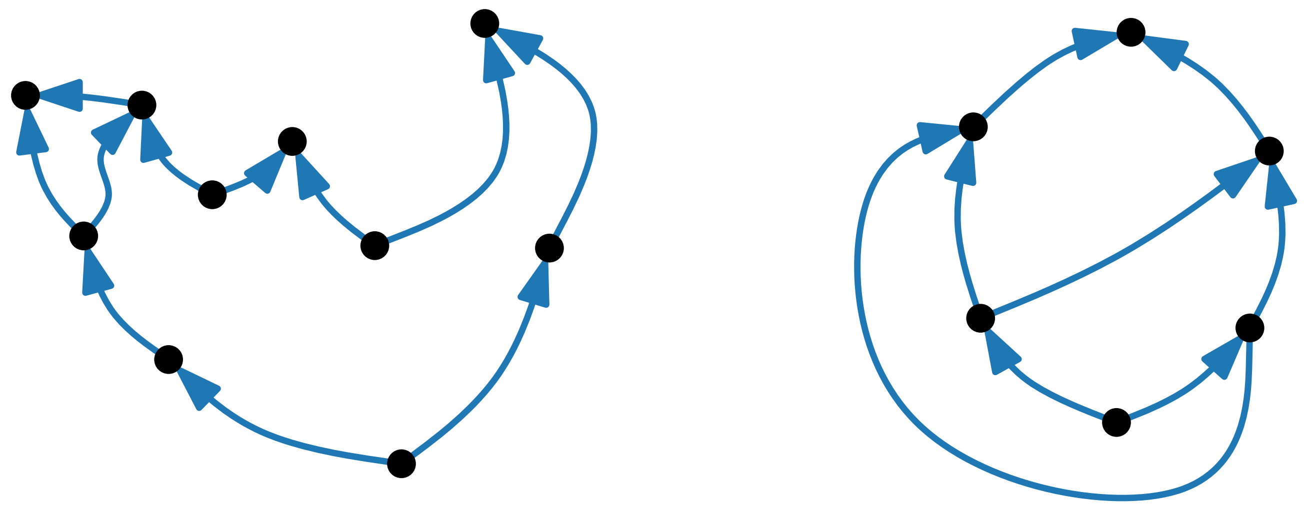

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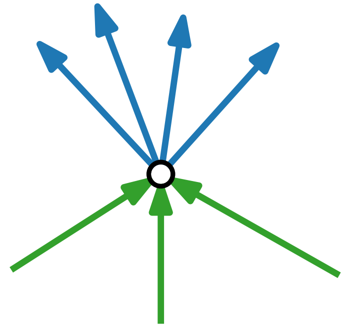

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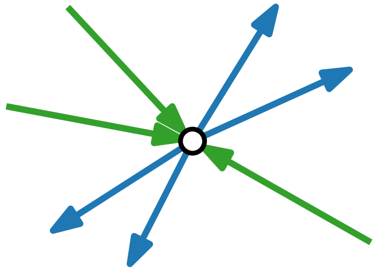
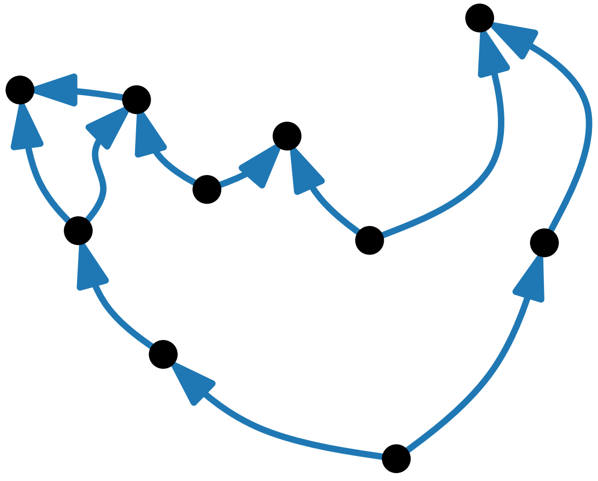
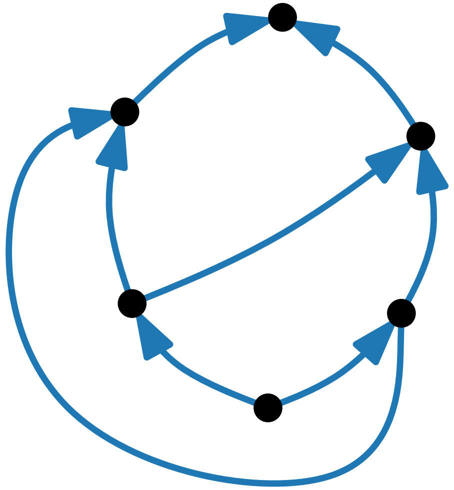


# Upward planarity – necessary conditions

- For a digraph $G$ to be upward planar, it has to be:
  - planar
  - acyclic
  - bimodal
- . . . but these conditions are *not sufficient*.





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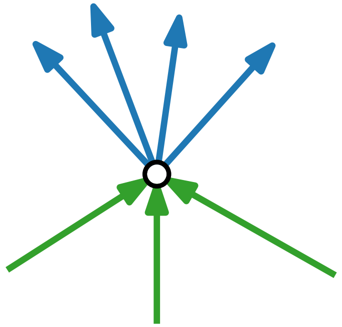

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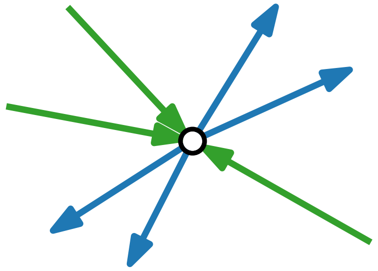

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**Theorem 1.** [Kelly 1987, Di Battista, Tamassia, 1988, see GD Ch. 6]
For a digraph $G$ the following statements are equivalent:

1. $G$ is upward planar.
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*Additionally:*
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*Additionally:*
Embedded such that $s$ and $t$ are on the outerface $f_0$.
*or:*
Edge $(s, t)$ exists.

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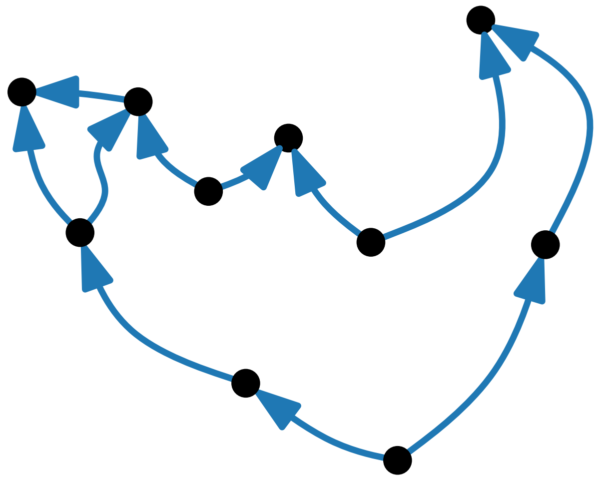

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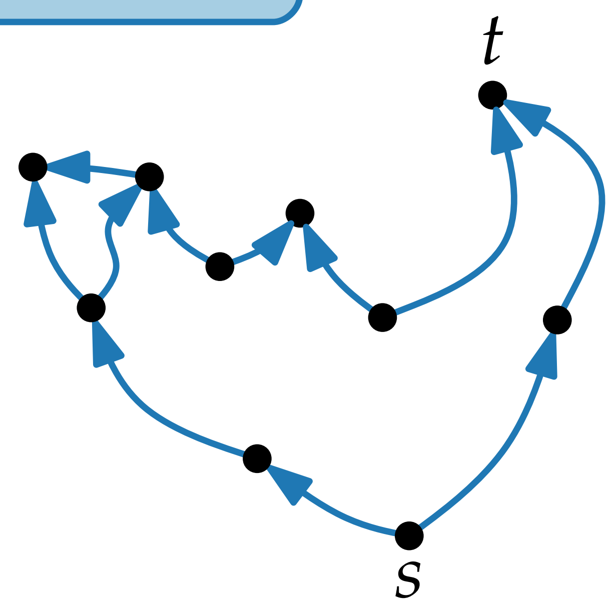

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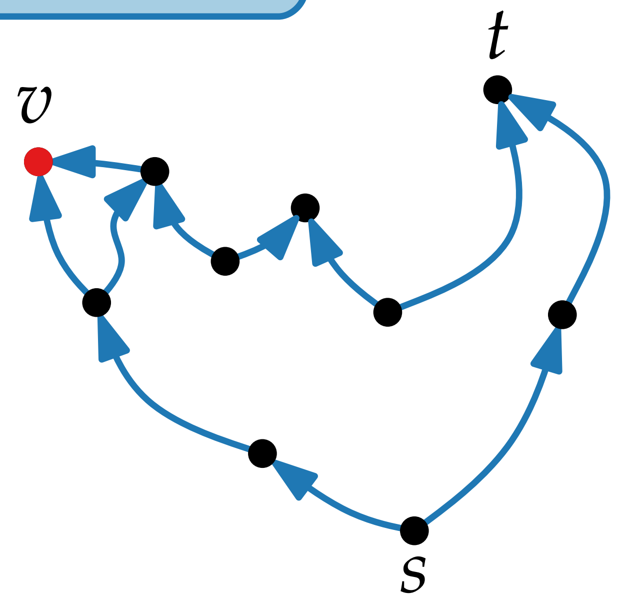

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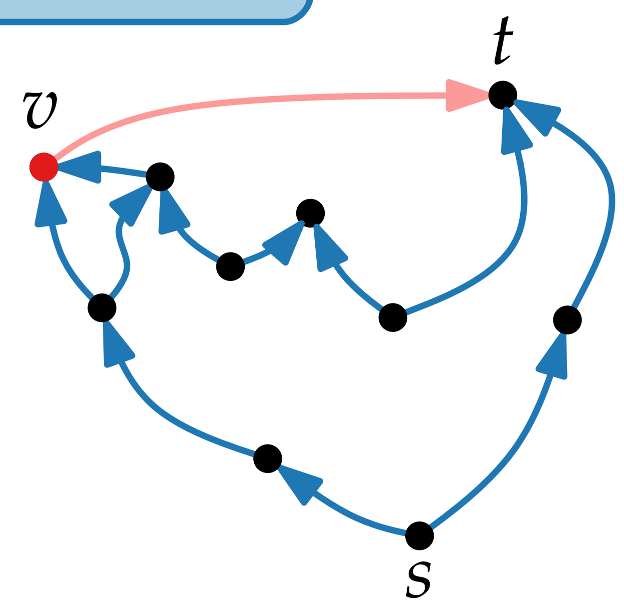

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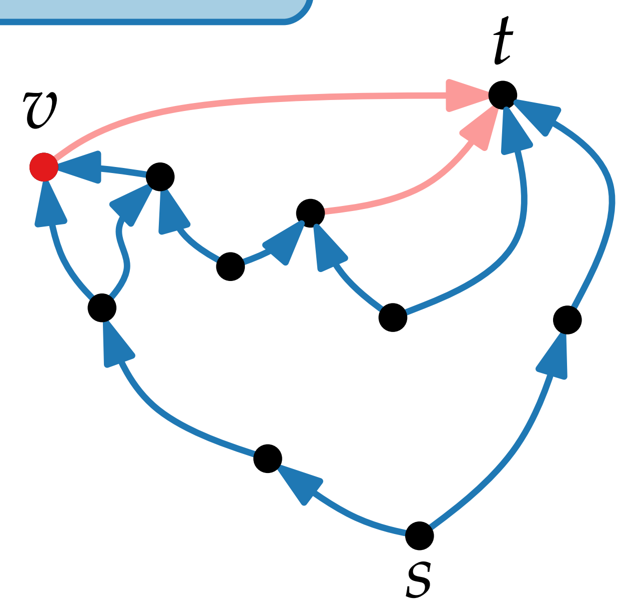

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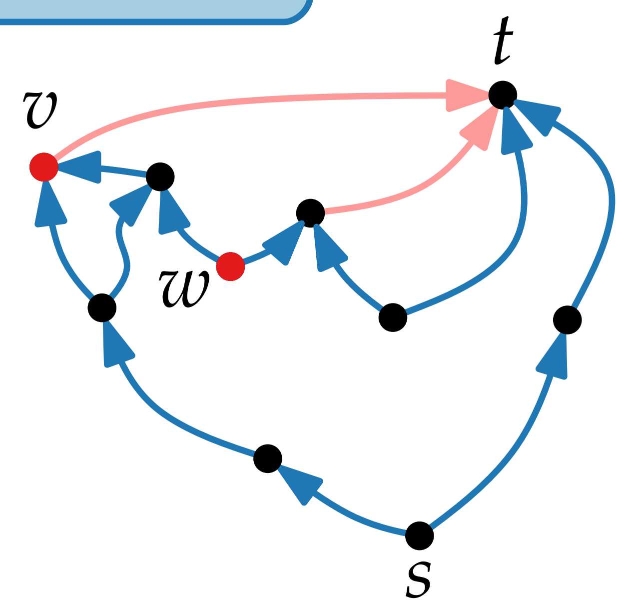

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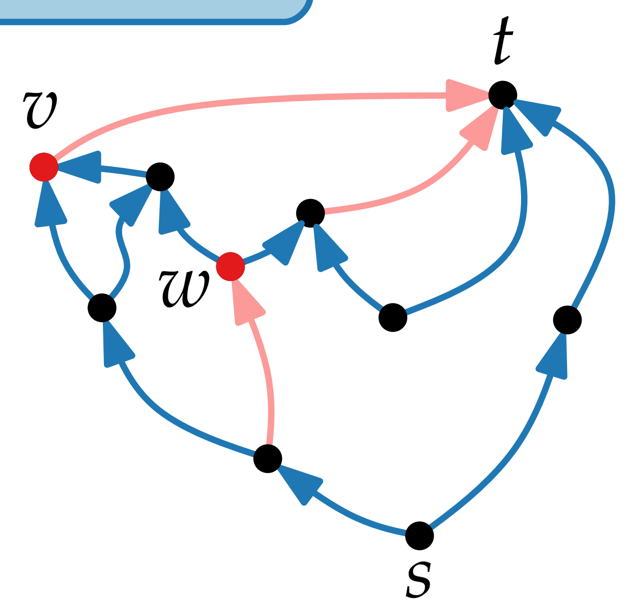

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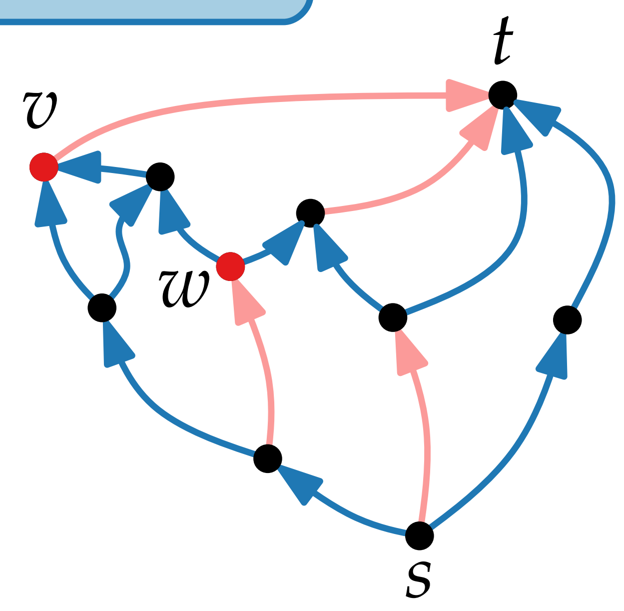

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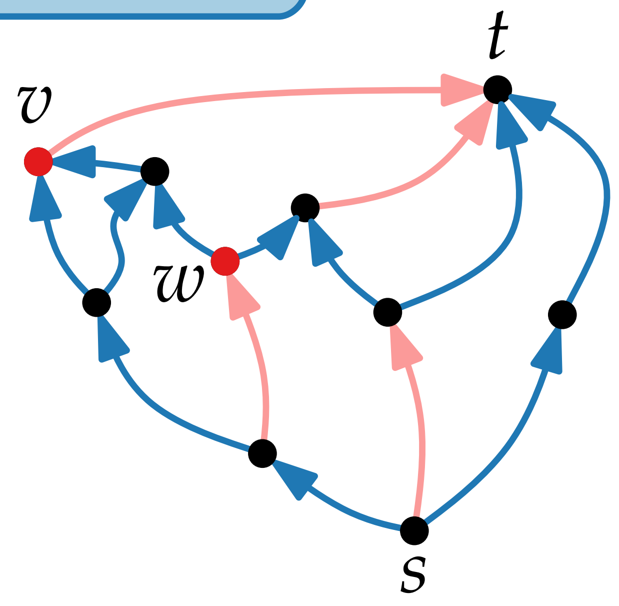

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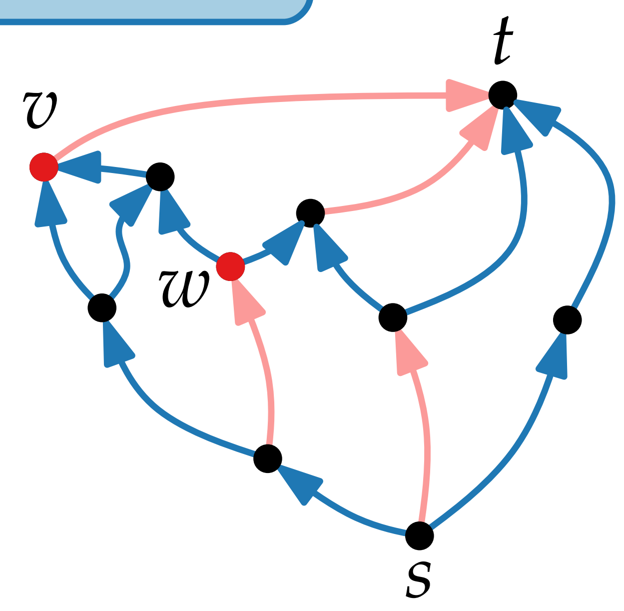
**Claim.**
Can draw in prespecified triangle.



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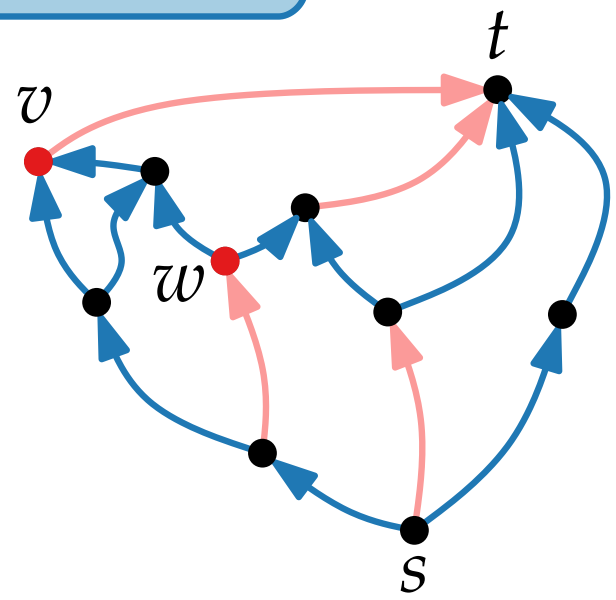


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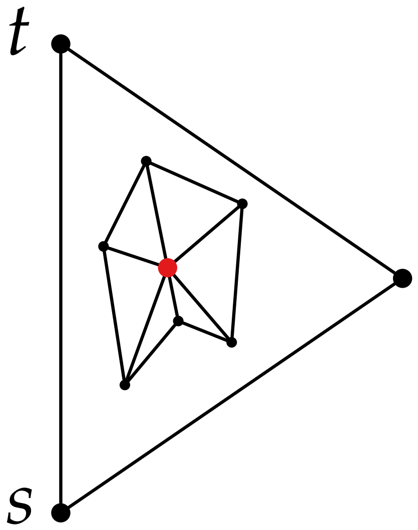

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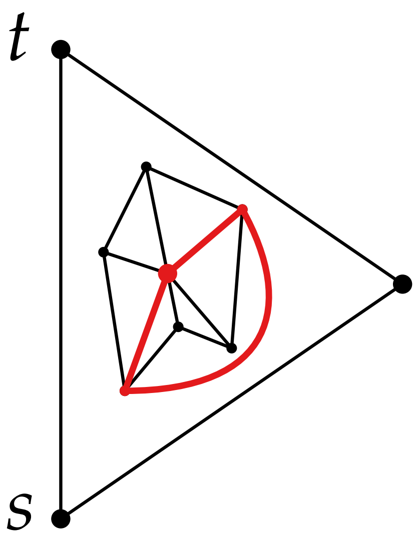


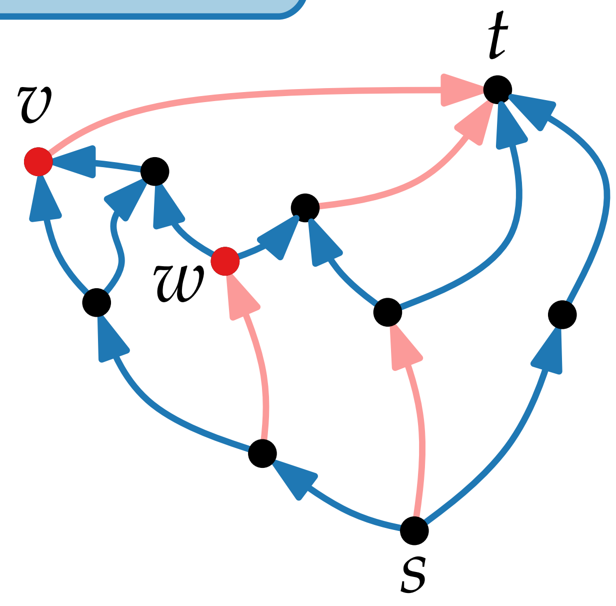

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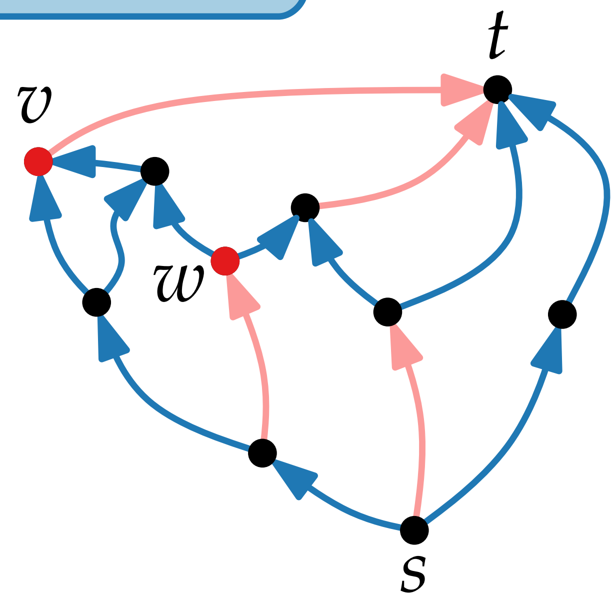


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Apply induction.

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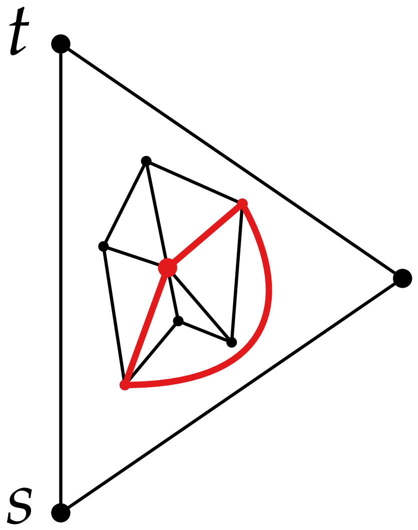

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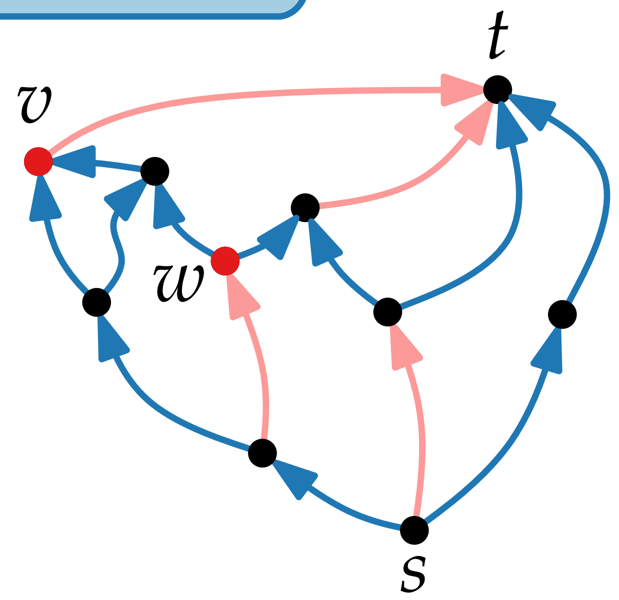


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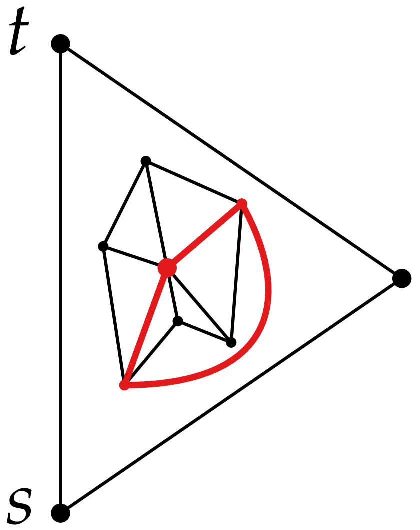


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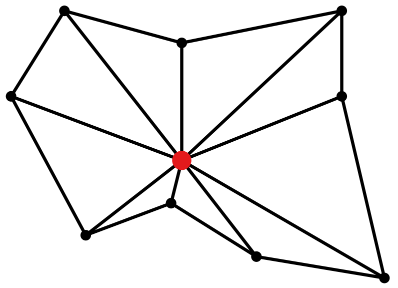

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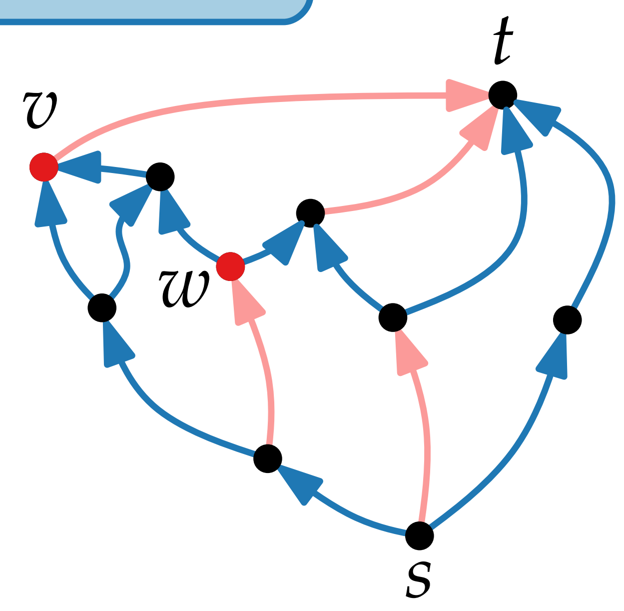


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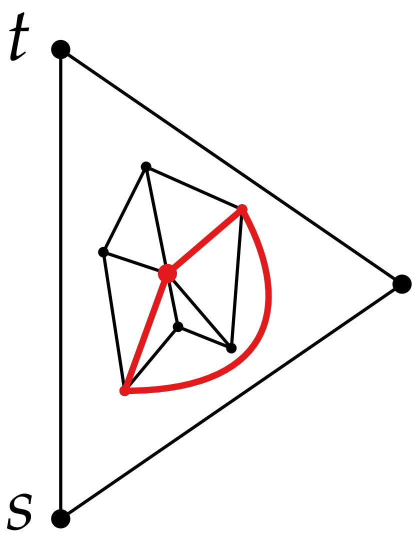


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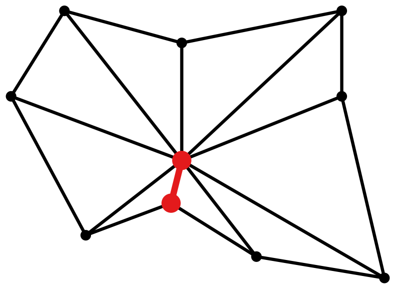

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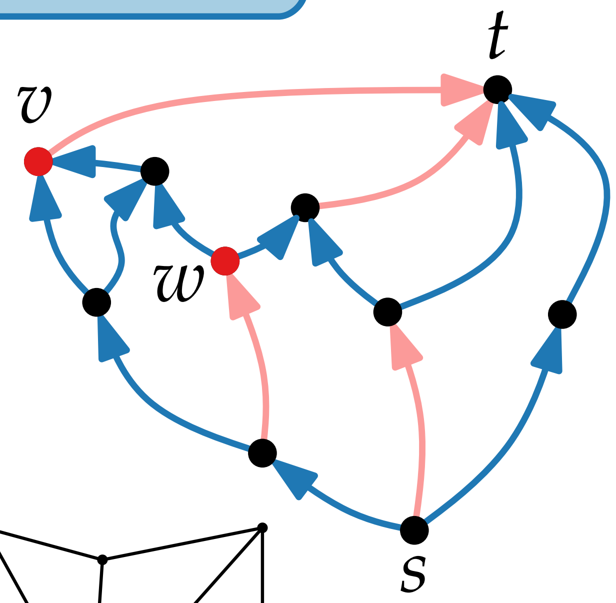
**Theorem 1.** [Kelly 1987, Di Battista, Tamassia, 1988, see GD Ch. 6]
For a digraph $G$ the following statements are equivalent:
1. $G$ is upward planar.
2. $G$ admits an upward planar straight-line drawing.
3. $G$ is the spanning subgraph of a planar $st$-digraph.

**Proof.**

**(2) $\Rightarrow$ (1)** By definition. **(1) $\Leftrightarrow$ (3)** Example:




**(3) $\Rightarrow$ (2)** Triangulate & construct drawing:

**Claim.**
Can draw in prespecified triangle. Apply induction.

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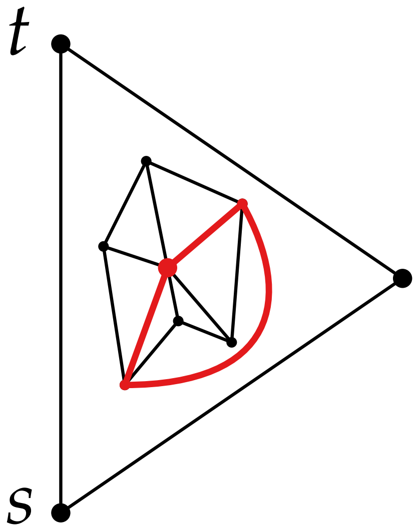


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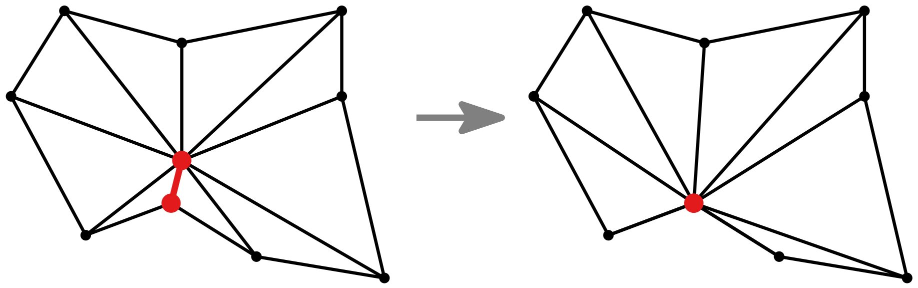

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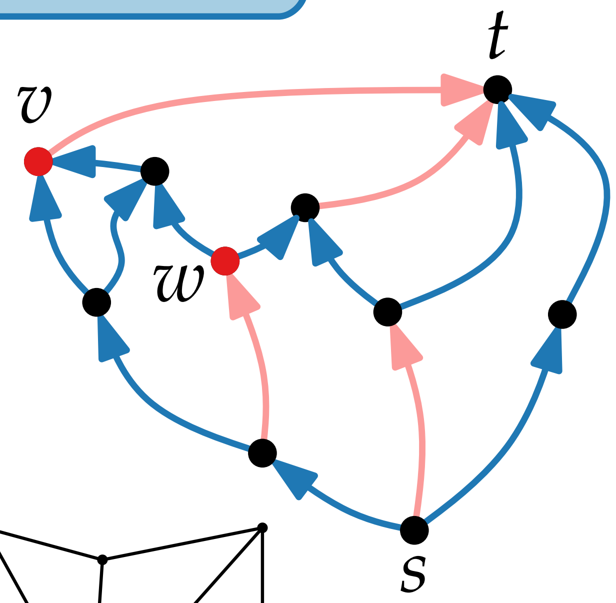


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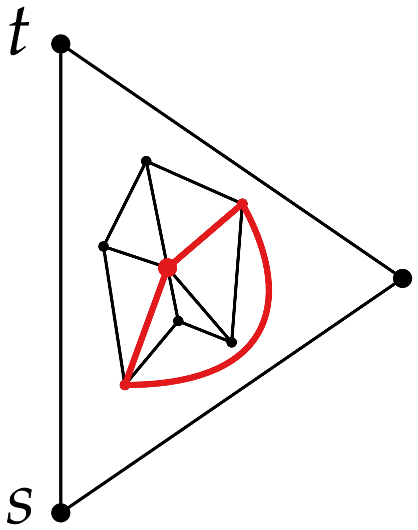


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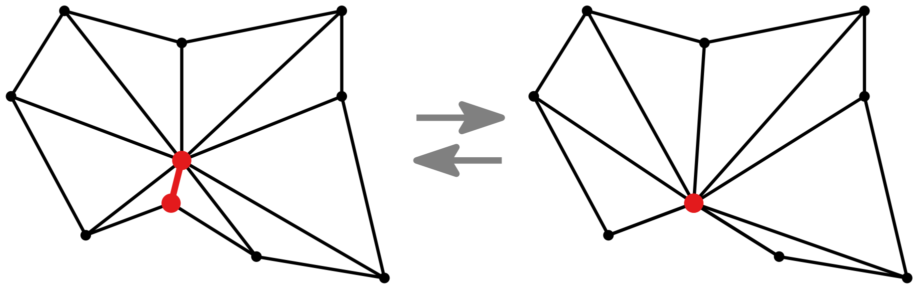

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**Theorem.** [Hutton, Libow, 1996]
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# The problem

**Fixed embedding upward planarity testing.**
Let $G = (V, E)$ be a plane digraph with the embedding given by the set of faces $F$ and the outer face $f_0$.
Test whether $G$ is upward planar (wrt to $F$, $f_0$).

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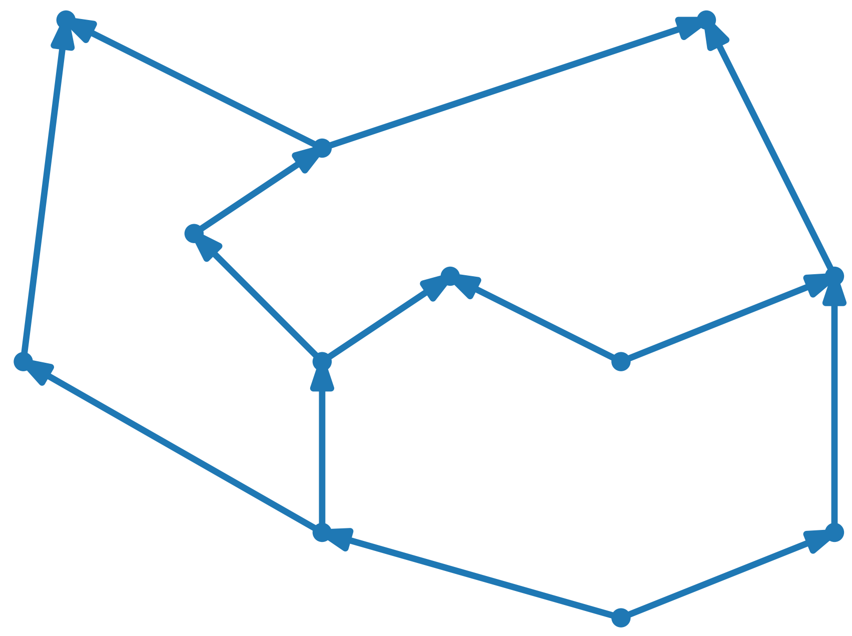
**Idea.**

- Find property that any upward planar drawing of $G$ satisfies.
- Formalise property.
- Find algorithm to test property.

# Angles, local sources & sinks

**Definitions.**



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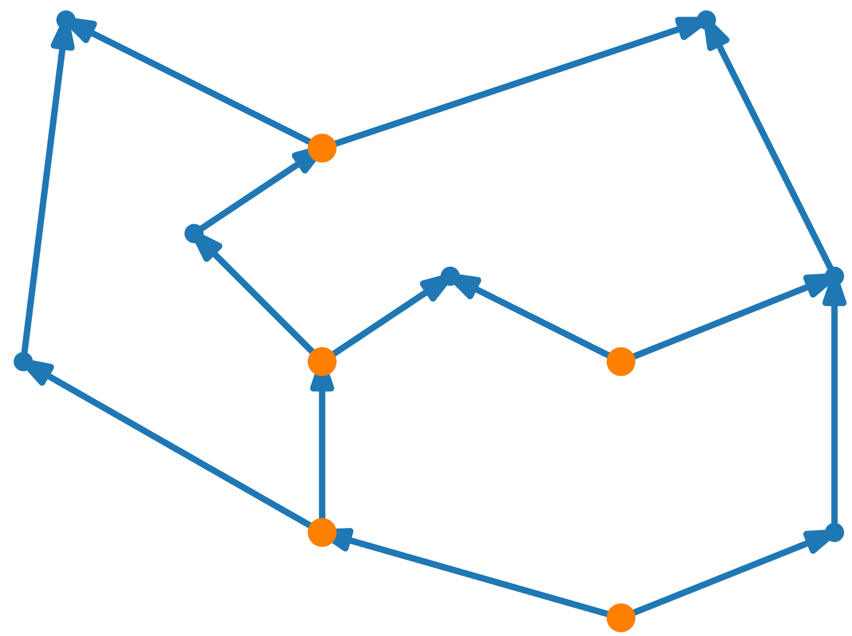

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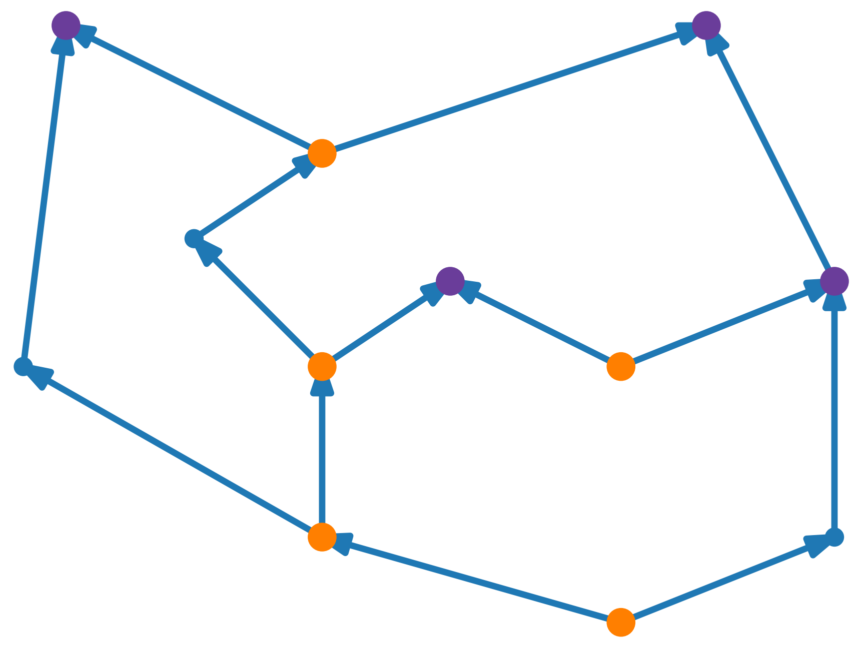

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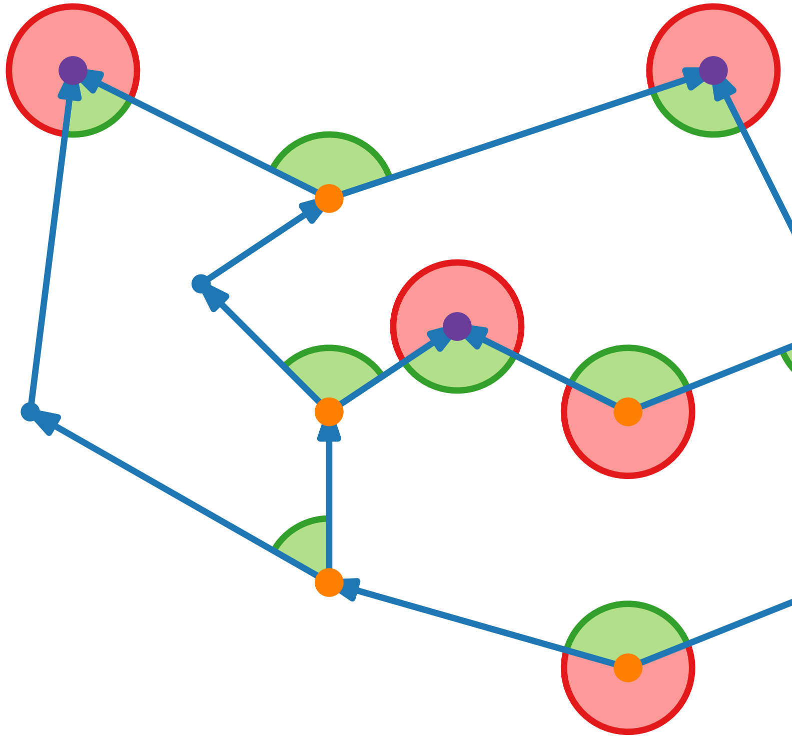

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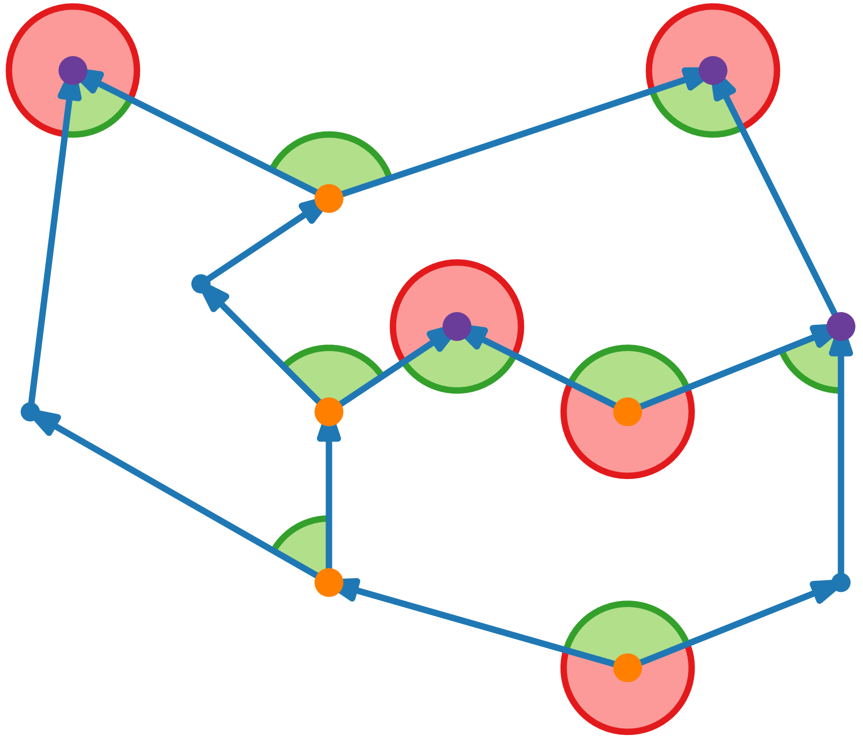

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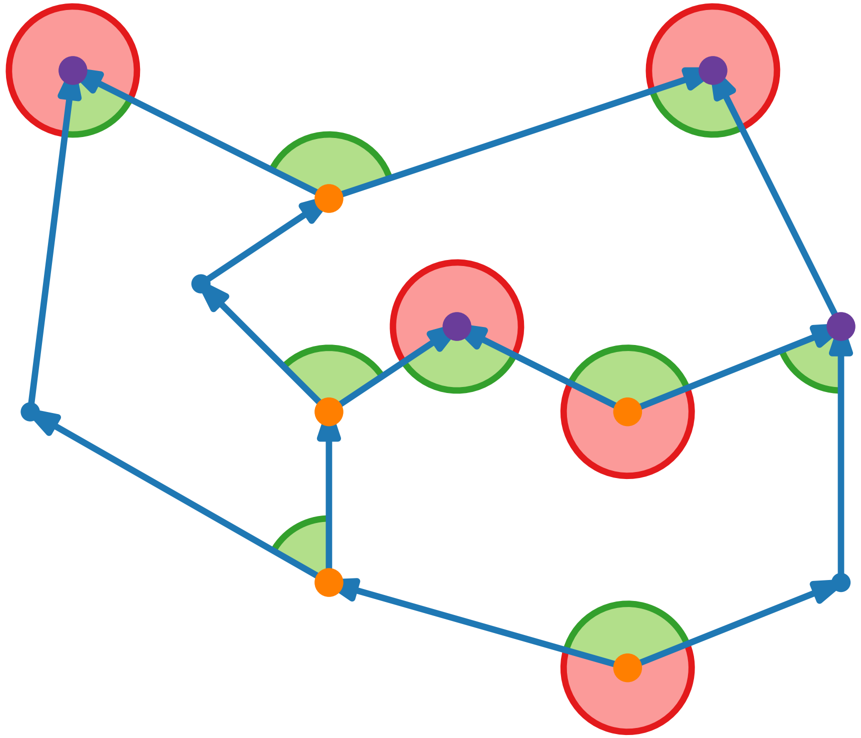

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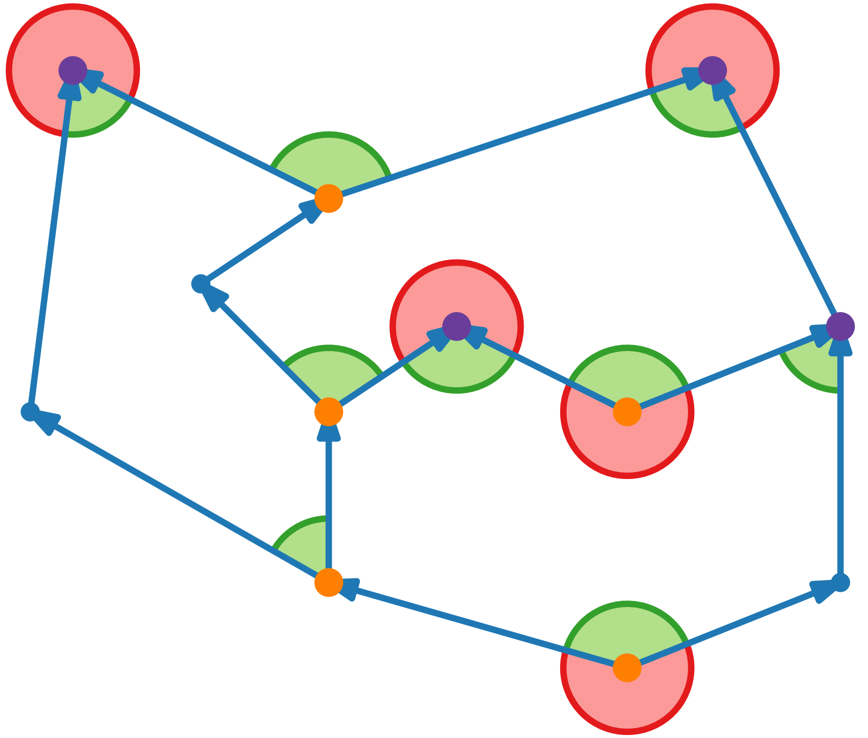
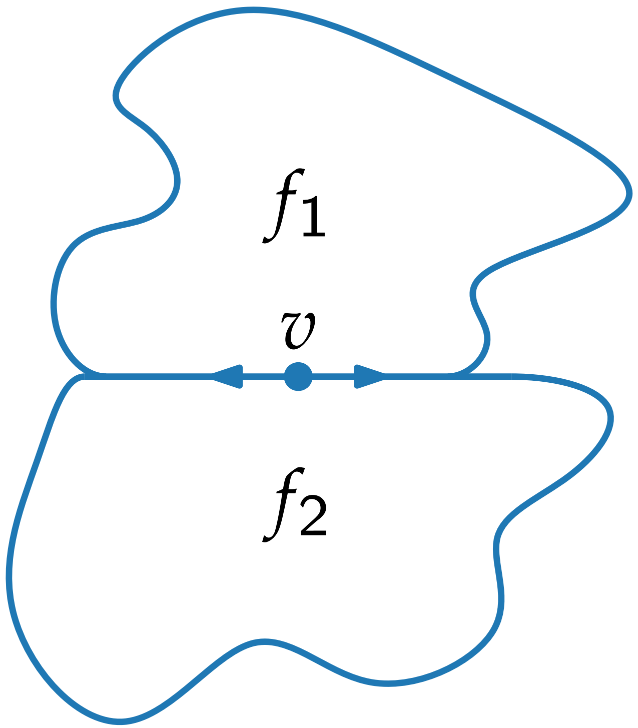


**Lemma 1.**
$L(f) + S(f) = 2A(f)$

# Assignment problem

- Vertex $v$ is a global source for $f_1$ and $f_2$.
- Has $v$ a **large** angle in $f_1$ or $f_2$?



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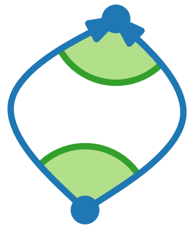

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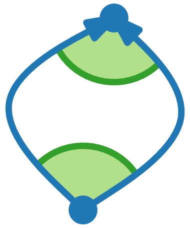
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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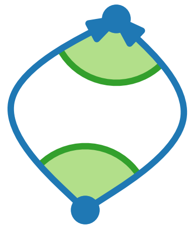
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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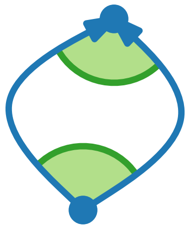
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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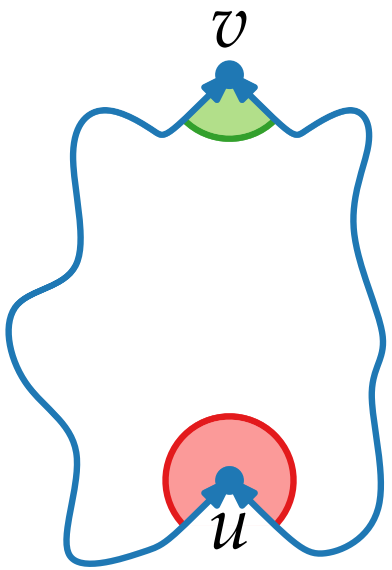

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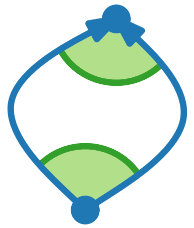
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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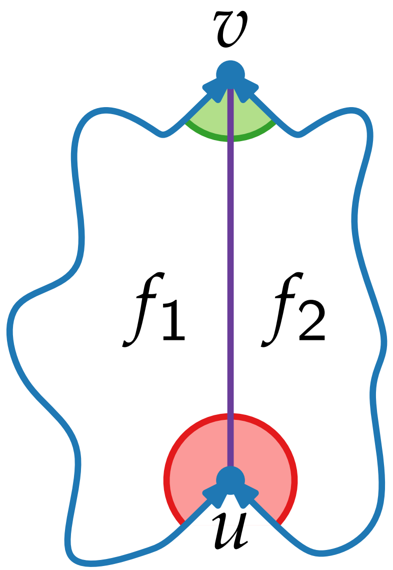

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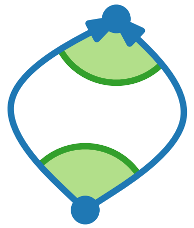

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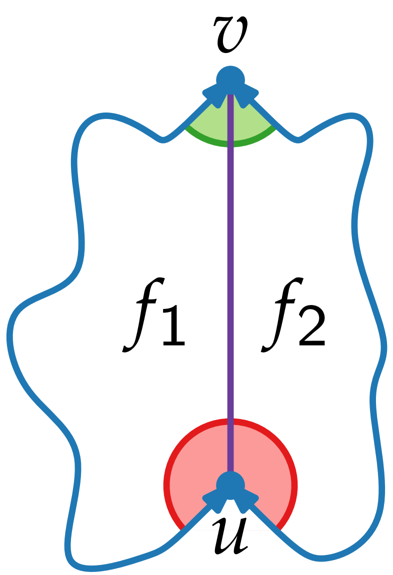


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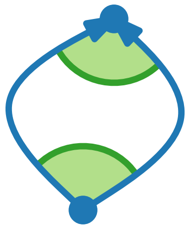

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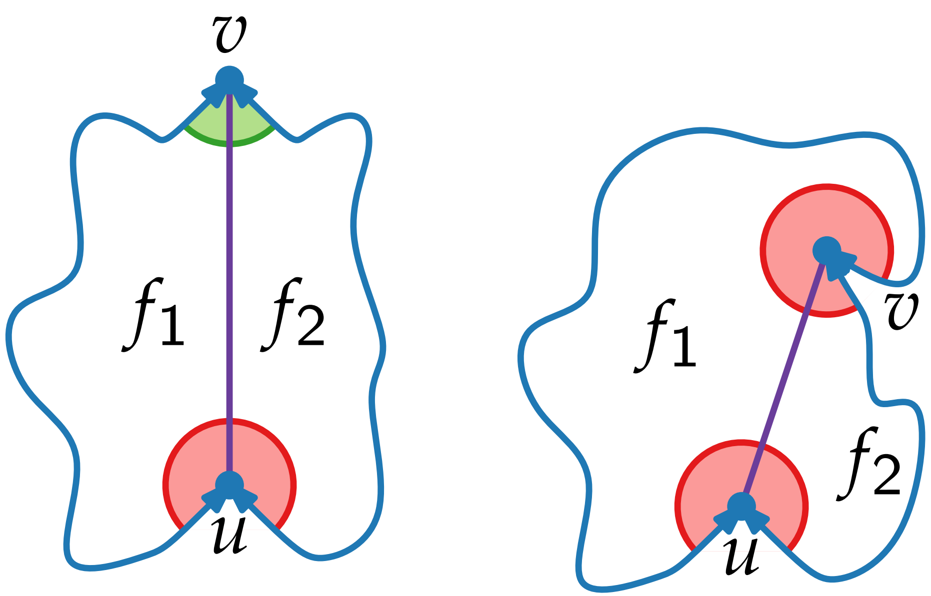


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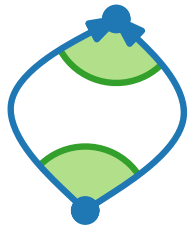
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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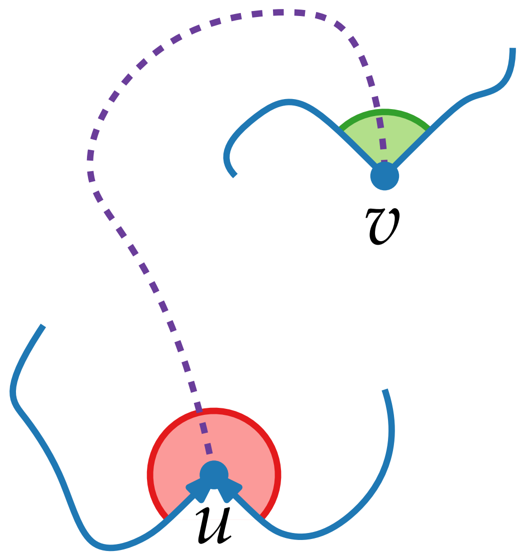

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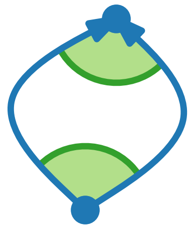
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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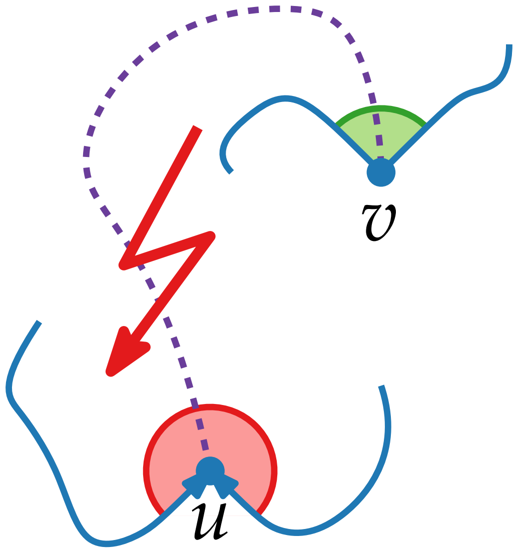

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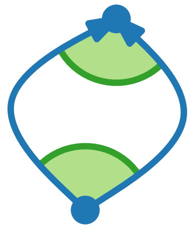
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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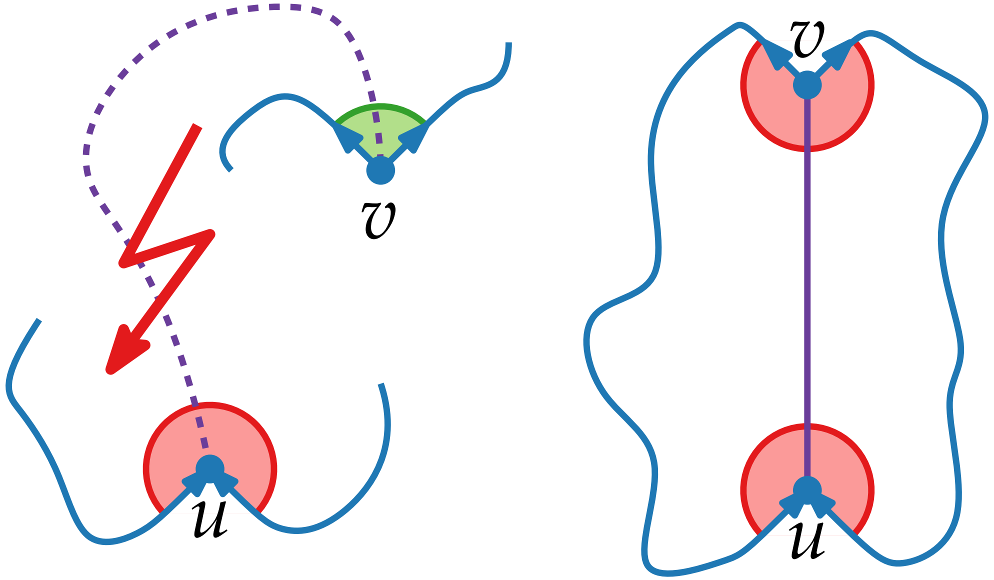


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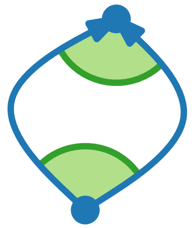

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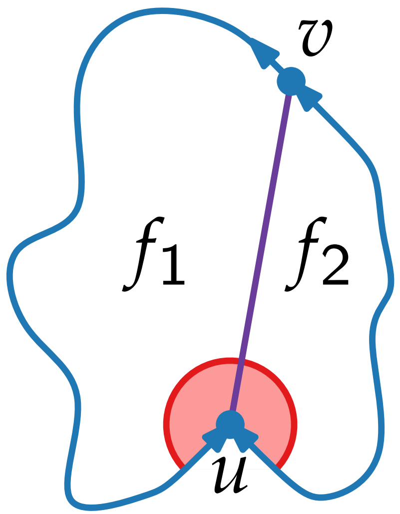


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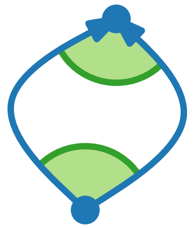
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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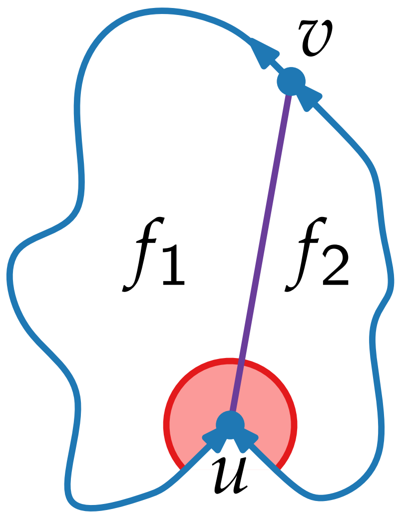


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- Otherwise "high" source $u$ exists.

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**Lemma 3.**
In every upward planar drawing of $G$ holds that

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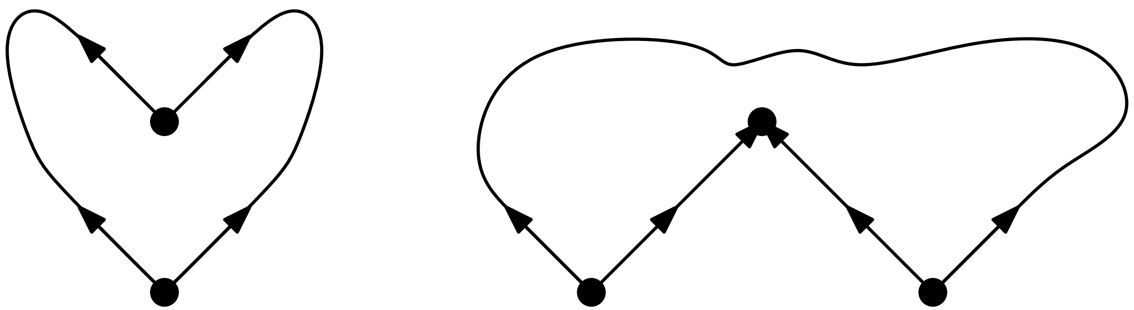

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# Assignment of large angles to faces

- Let $S$ and $T$ be the sets of sources and sinks, respectively.

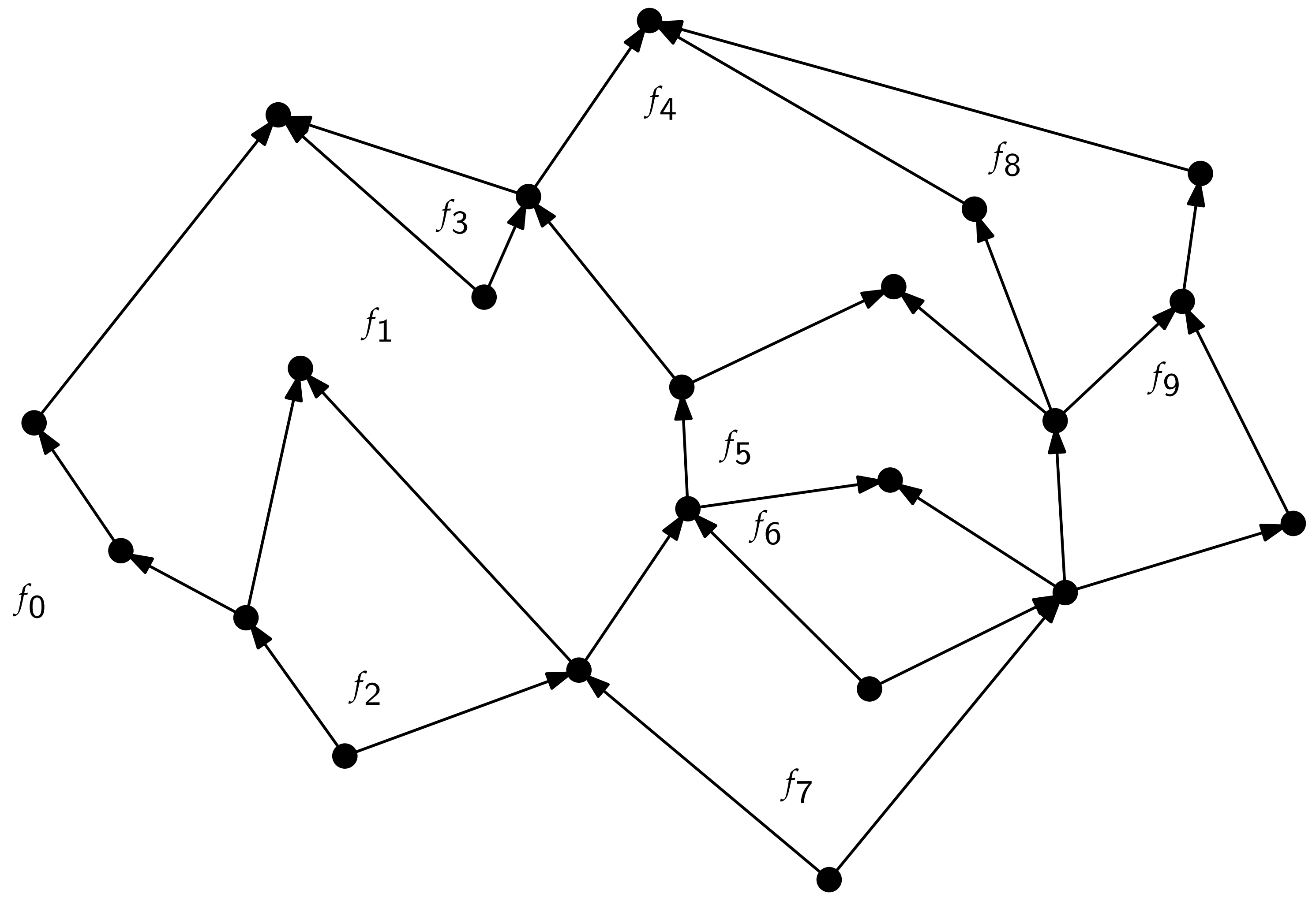
**Definition.**
A **consistent assignment** $\Phi\colon S \cup T \to F$ is a mapping where

$$\Phi\colon v \mapsto \text{ incident face, where } v \text{ forms large angle}$$

such that

$$|\Phi^{-1}(f)| = L(f) = \begin{cases} A(f) - 1 & \text{if } f \neq f_0, \\ A(f) + 1 & \text{if } f = f_0. \end{cases}$$

# Example of angle to face assignment



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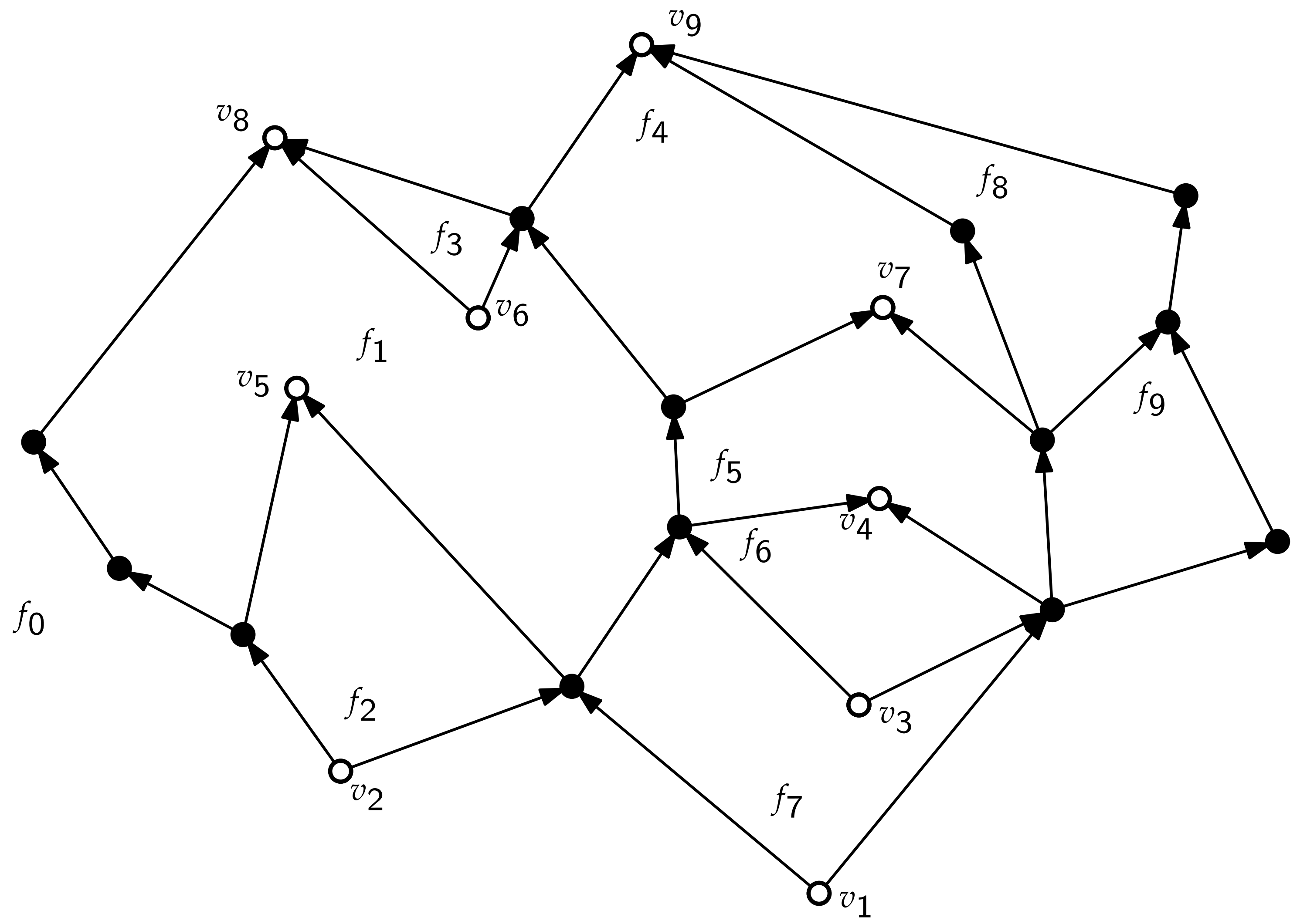


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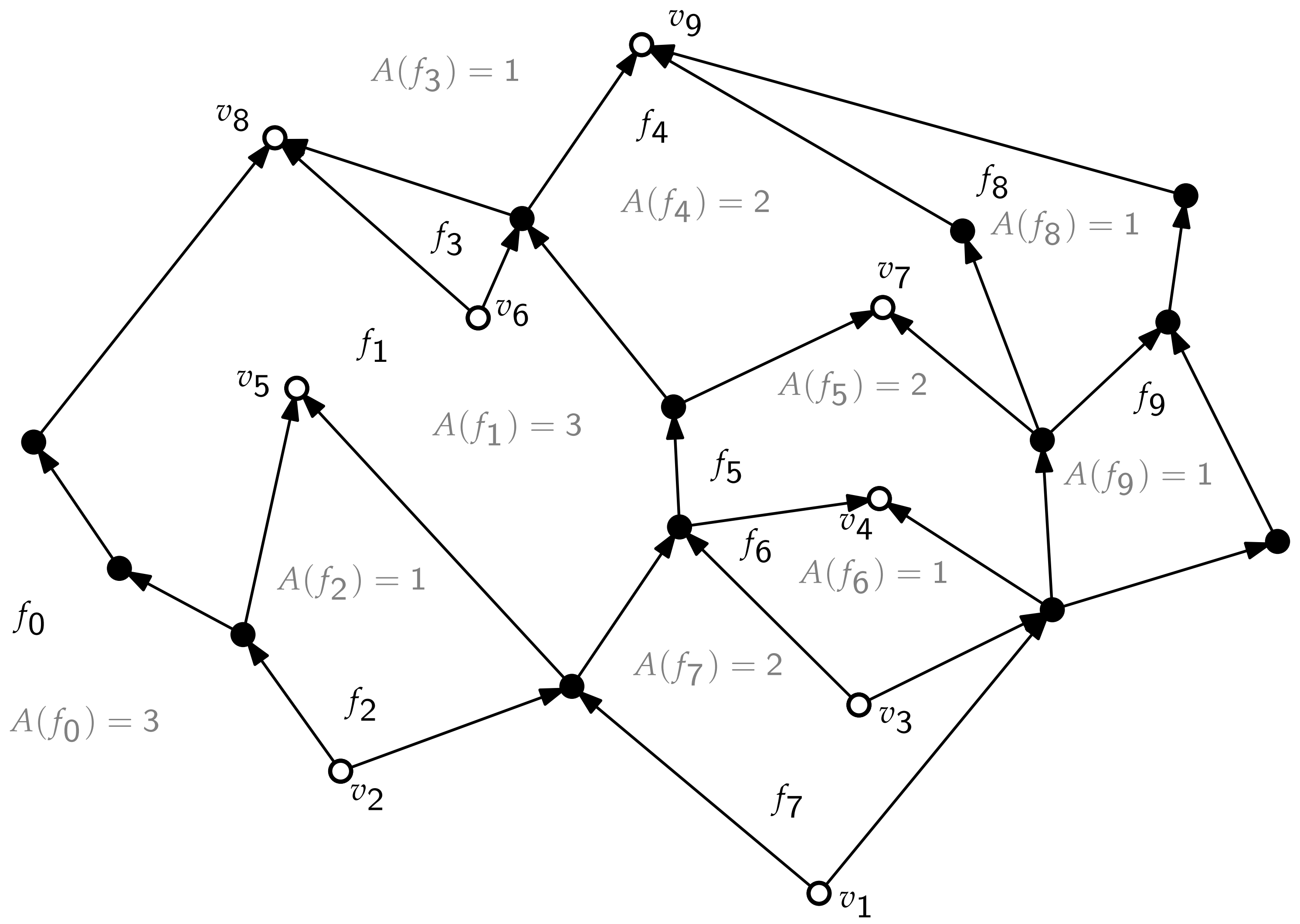


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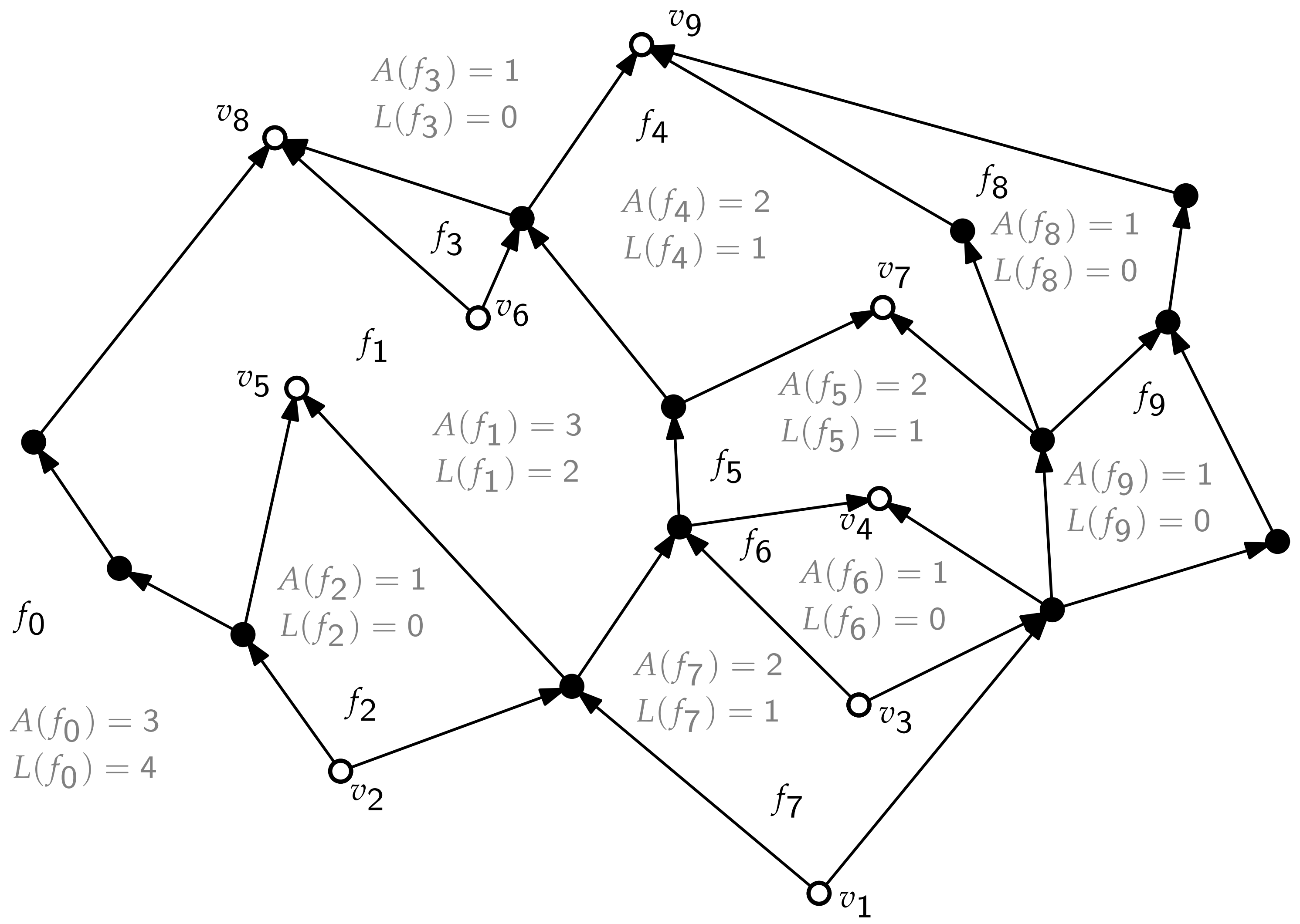


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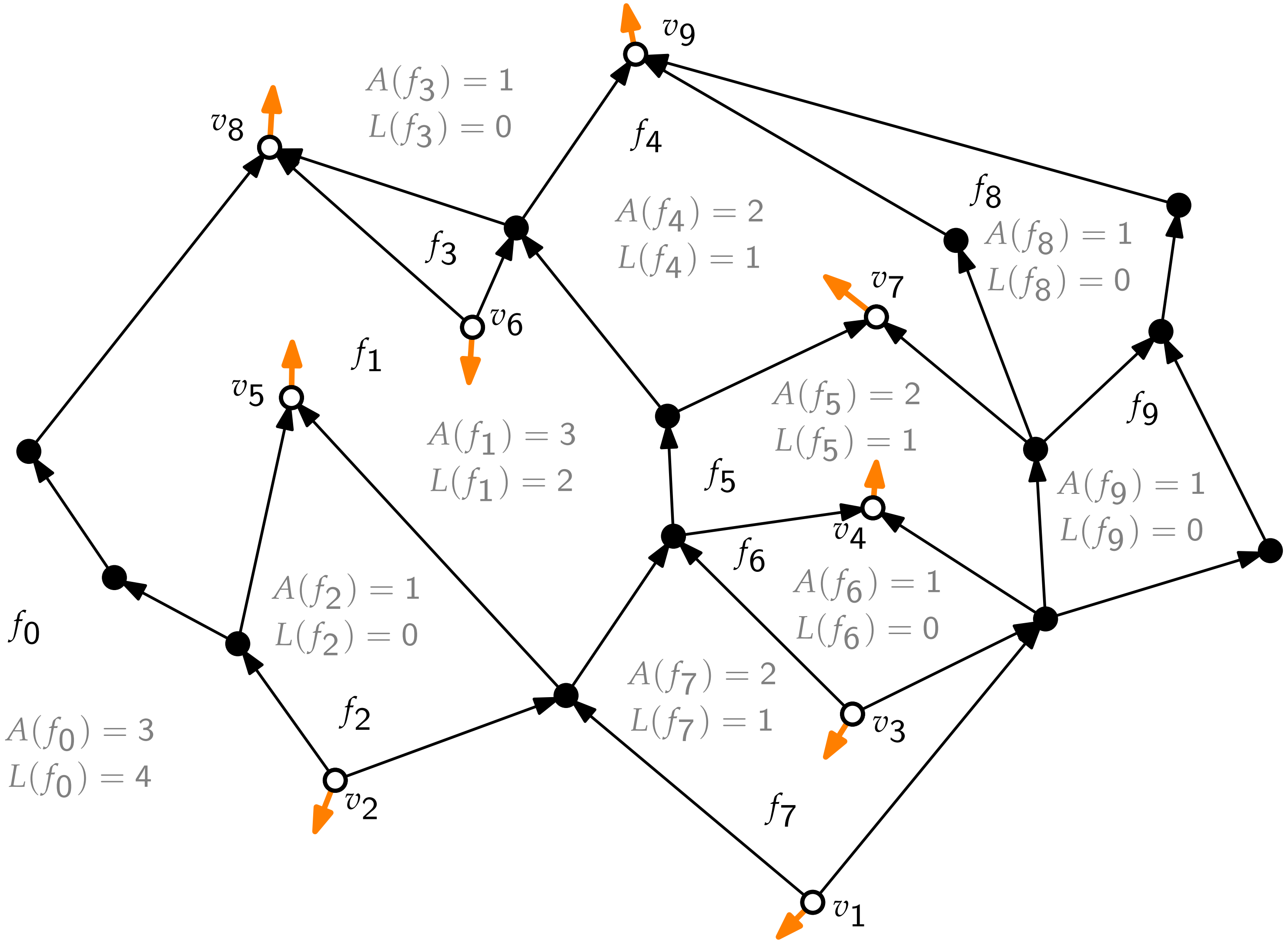


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# Result characterisation

**Theorem 3.**
Let $G = (V, E)$ be an acyclic plane digraph with embedding given by $F, f_0$.
Then $G$ is upward planar (respecting $F$, $f_0$) if and only if $G$ is bimodal and there exists consistent assignment $\Phi$.

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**Proof.**

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$\Leftarrow$: Idea:

- Construct planar st-digraph that is supergraph of $G$.
- Apply equivalence from Theorem 1.

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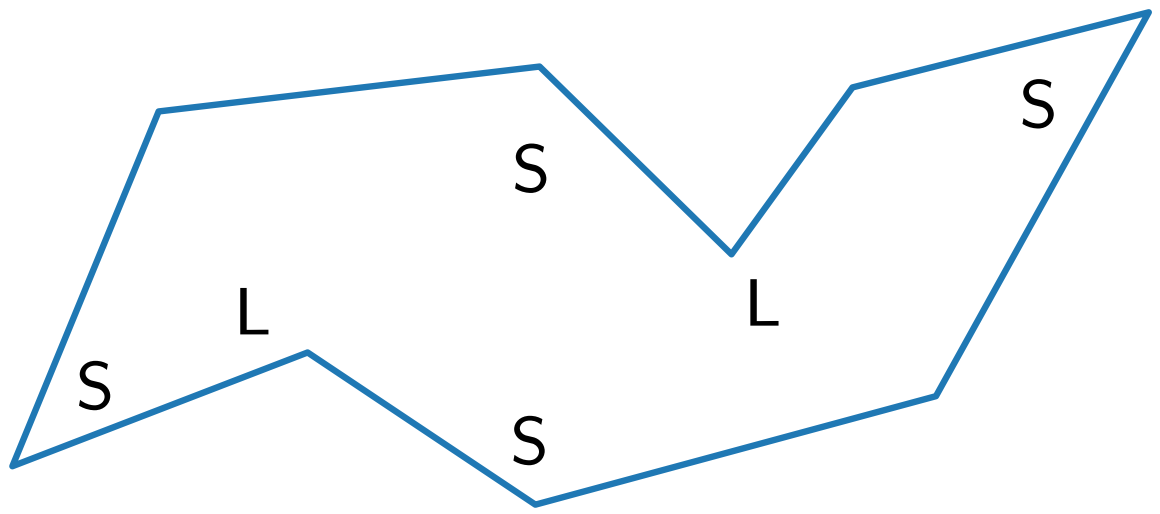

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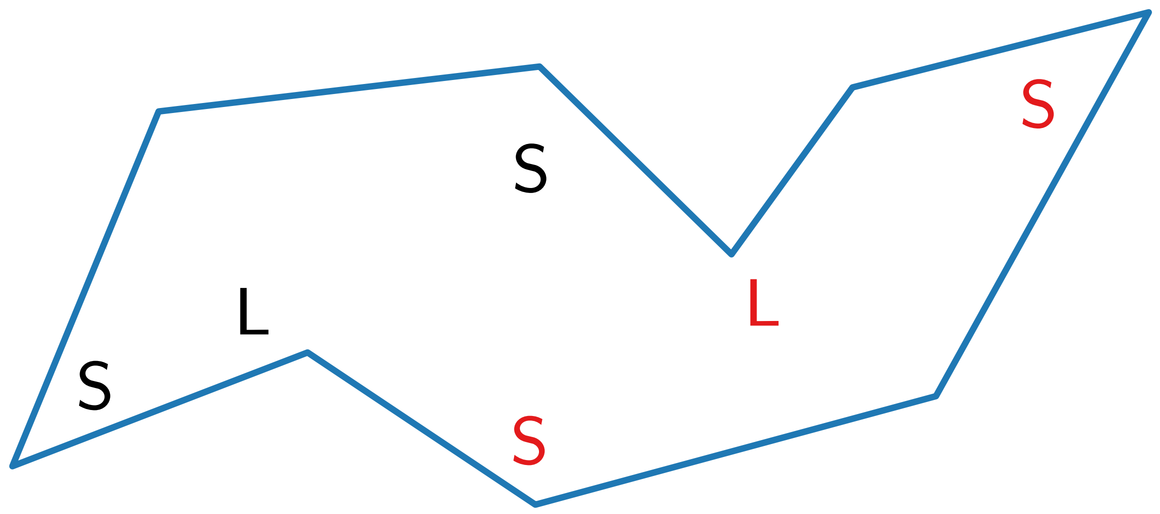

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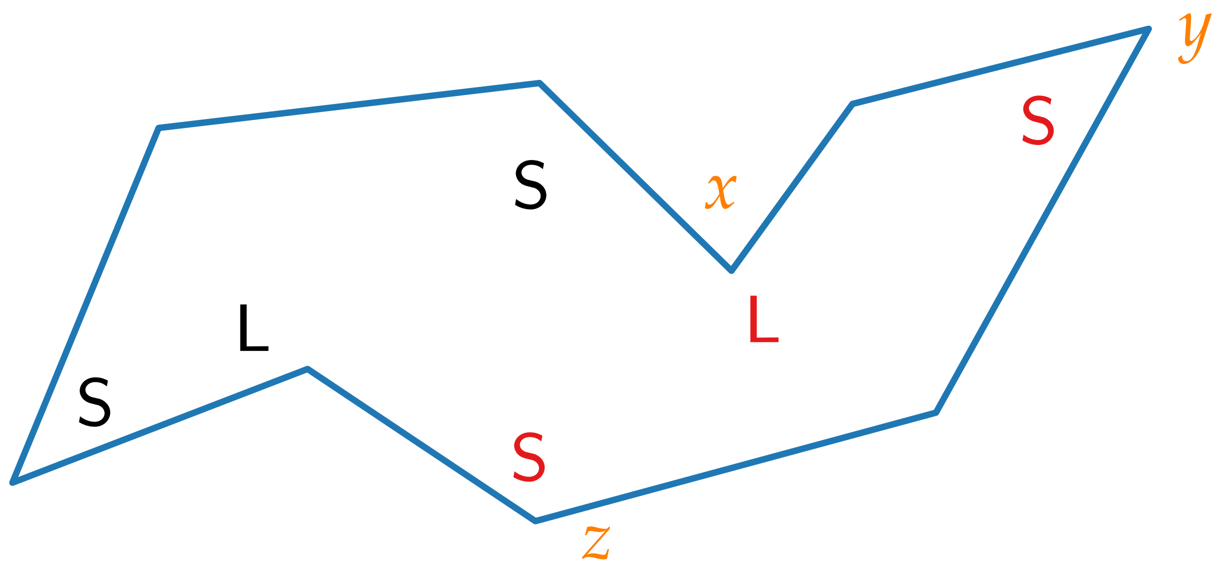

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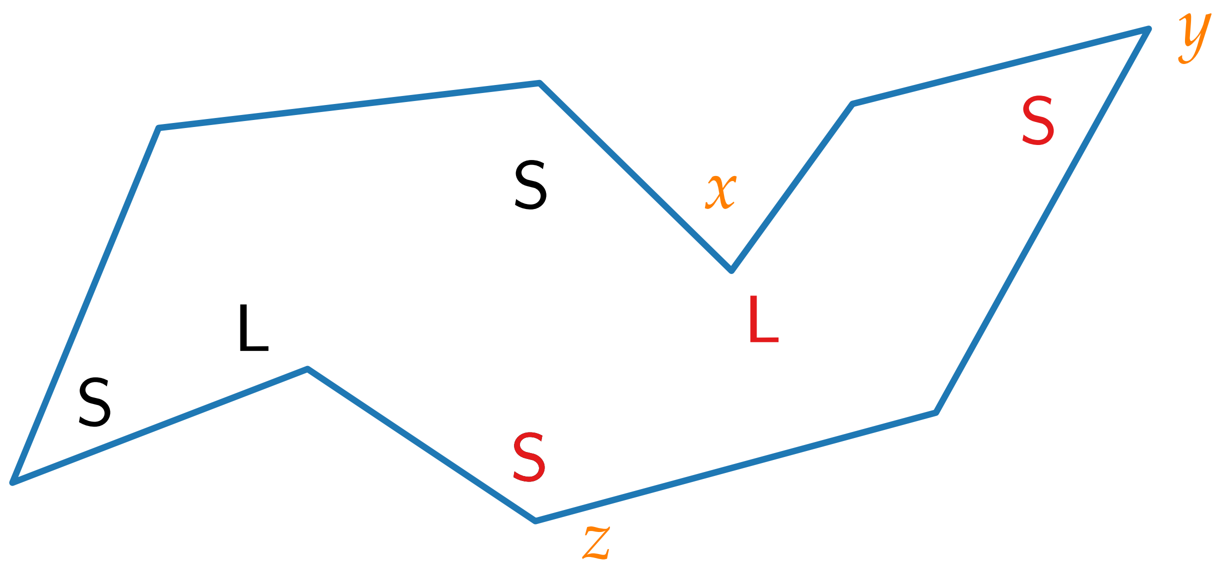

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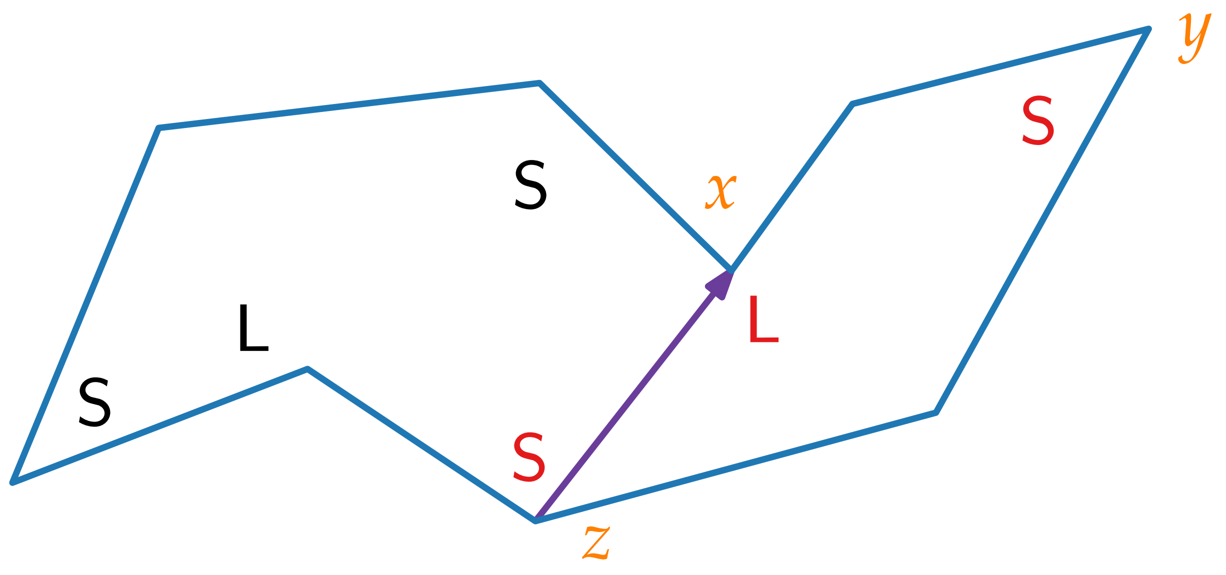

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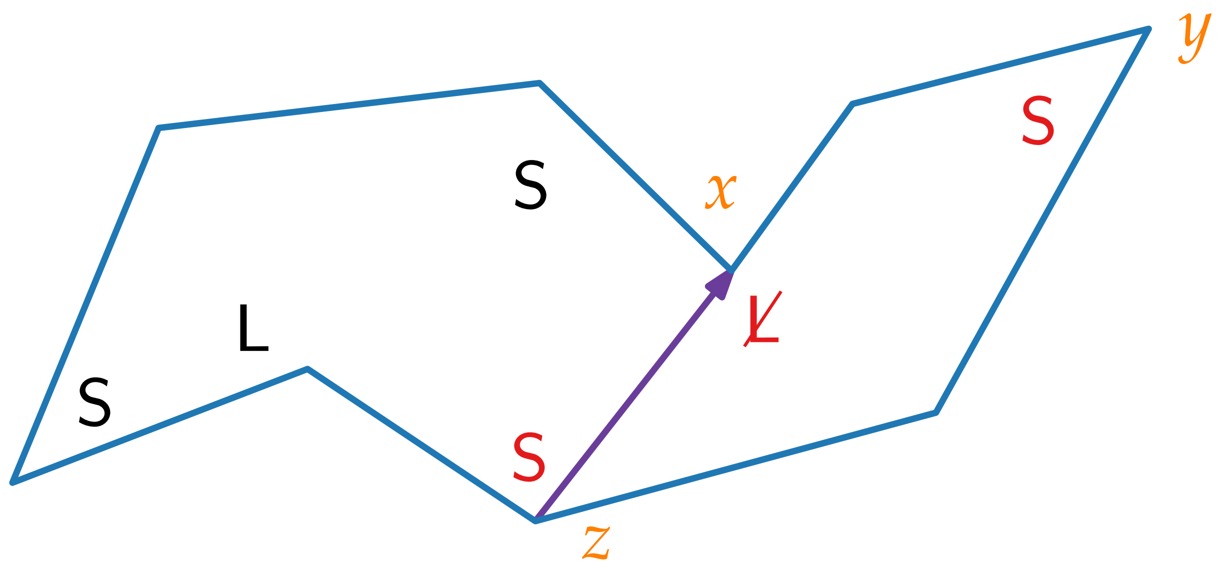

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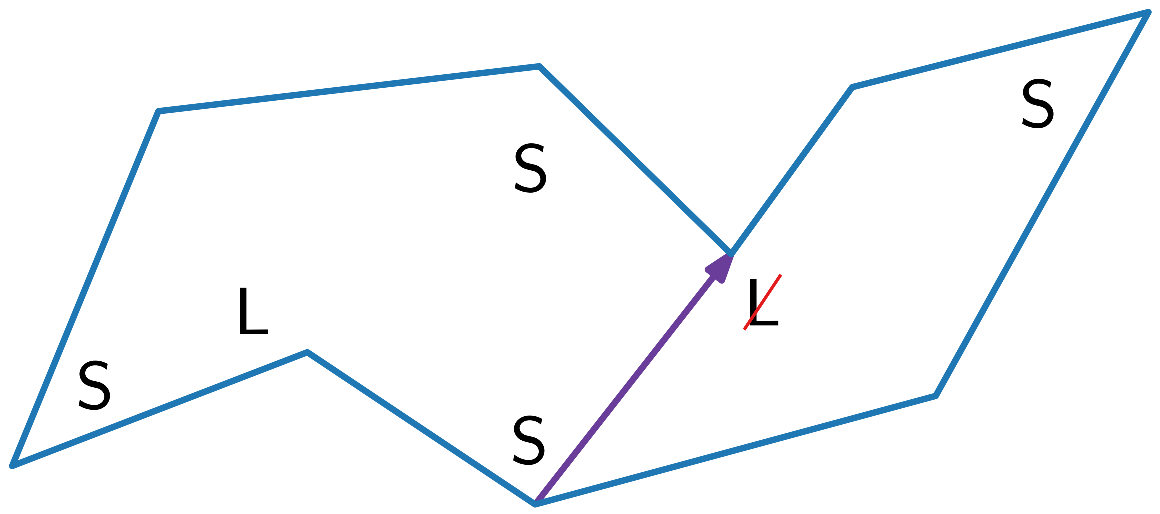

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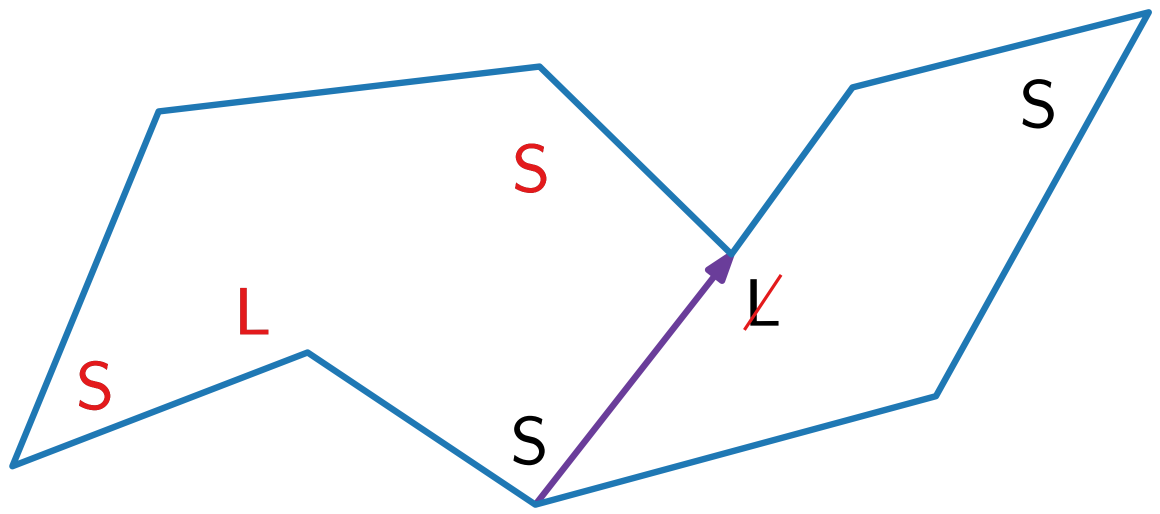

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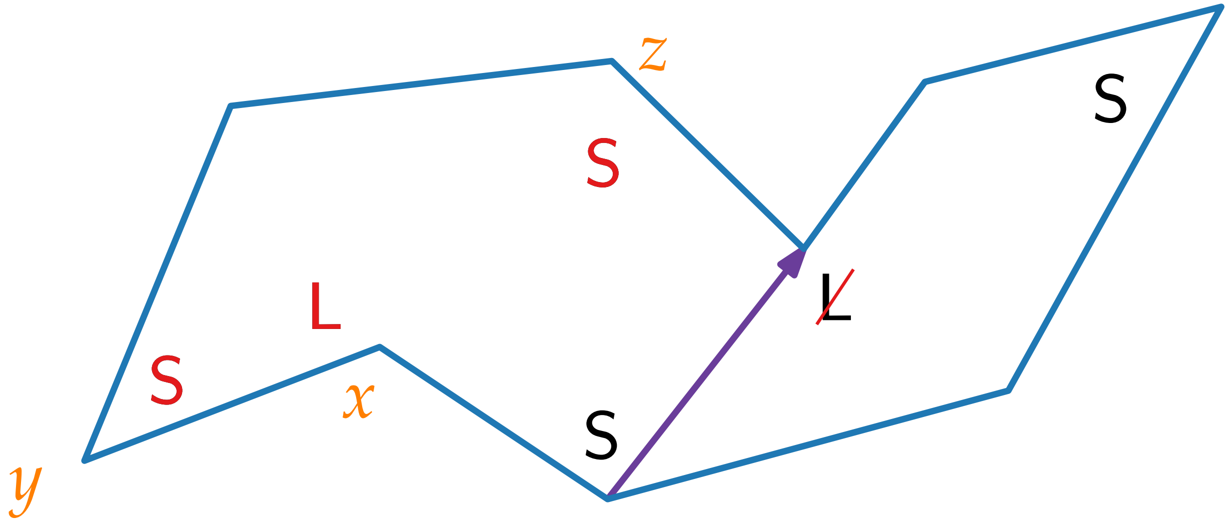

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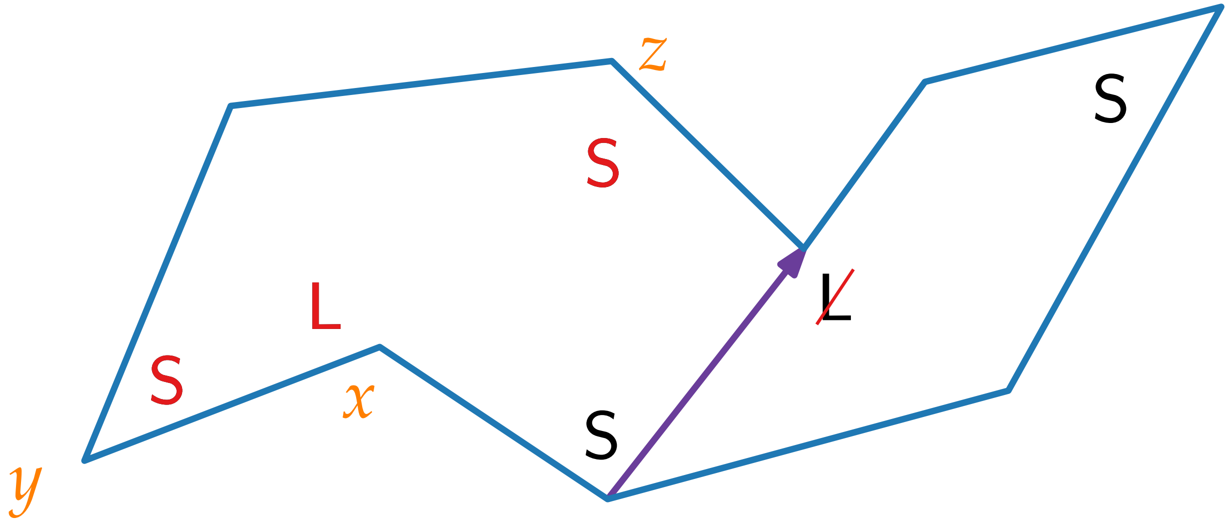

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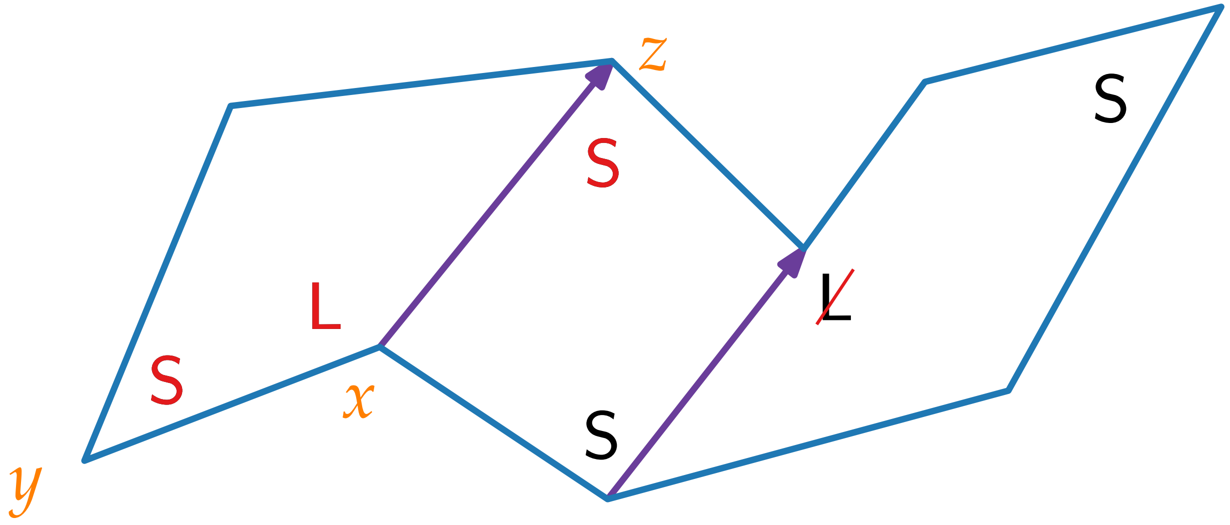

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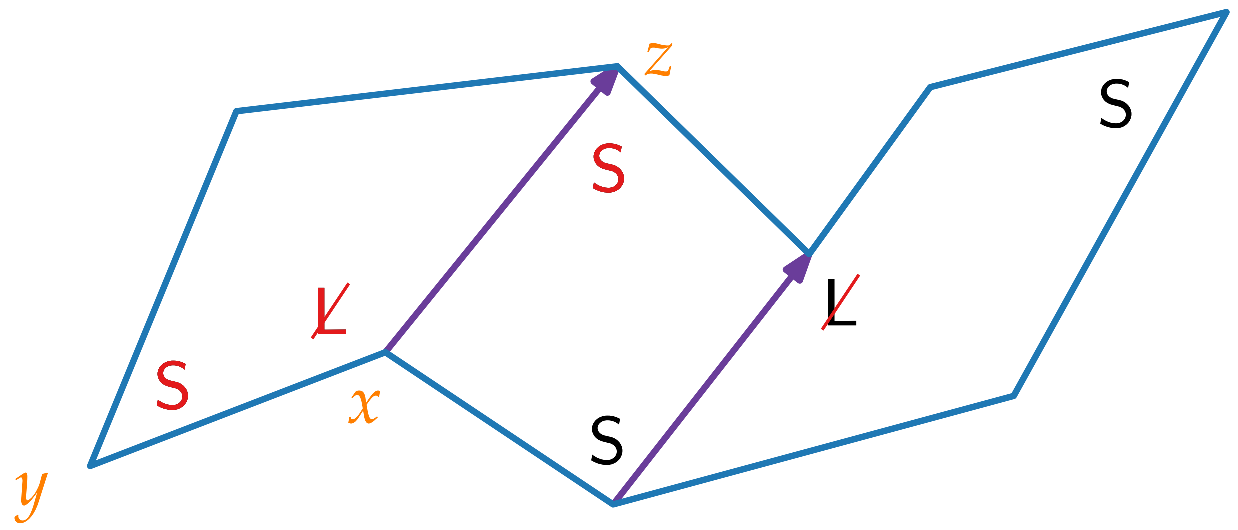

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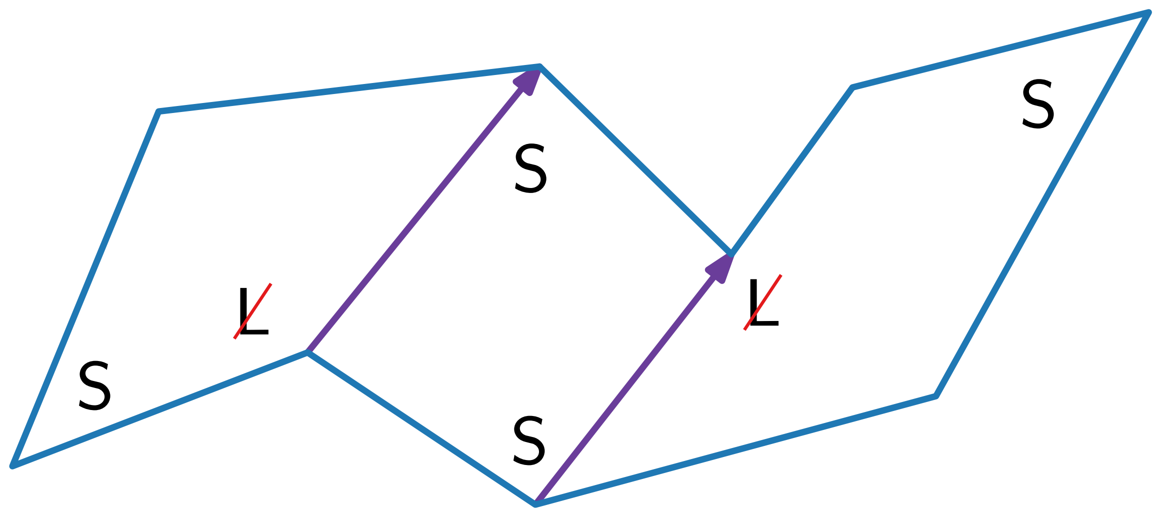

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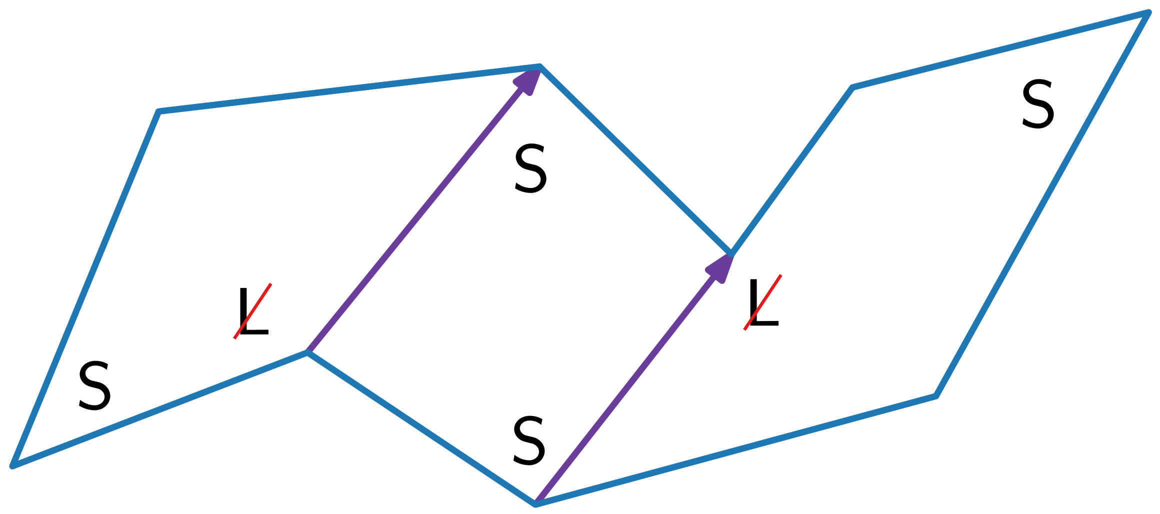

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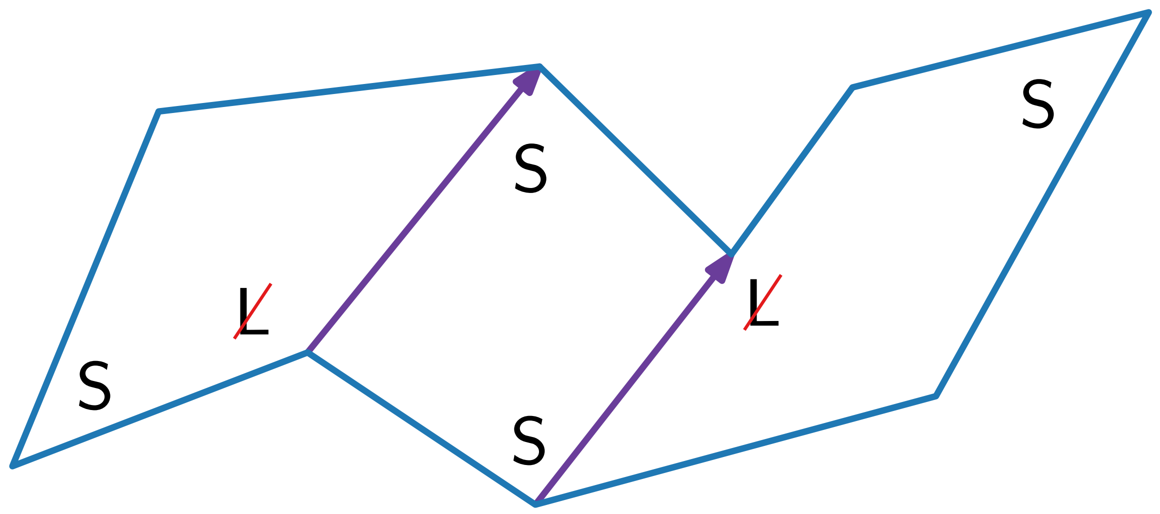
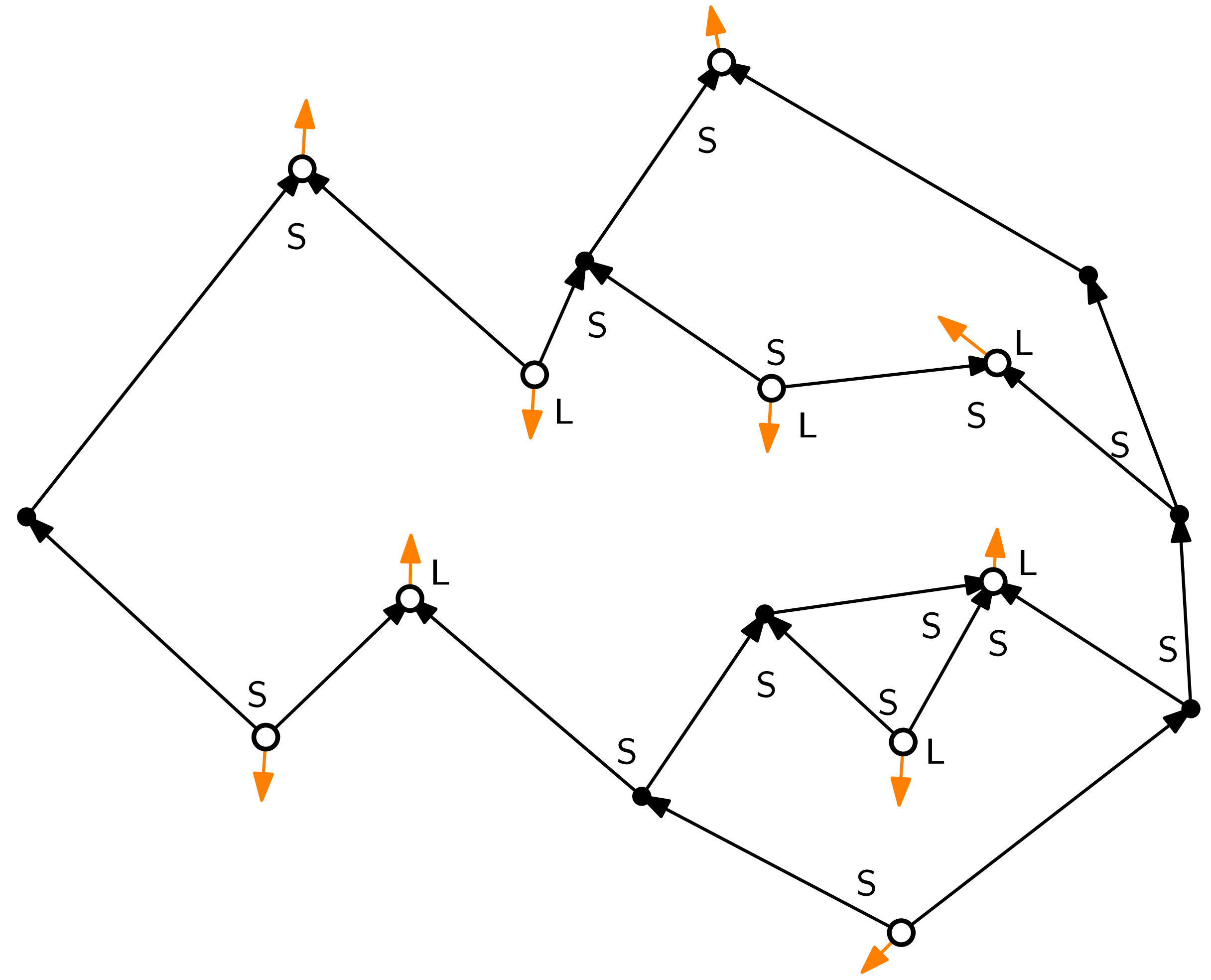



- Refine all faces. $\Rightarrow$ $G$ is contained in a planar st-digraph.
- Planarity, acyclicity, bimodality are invariants under construction.

# Refinement example



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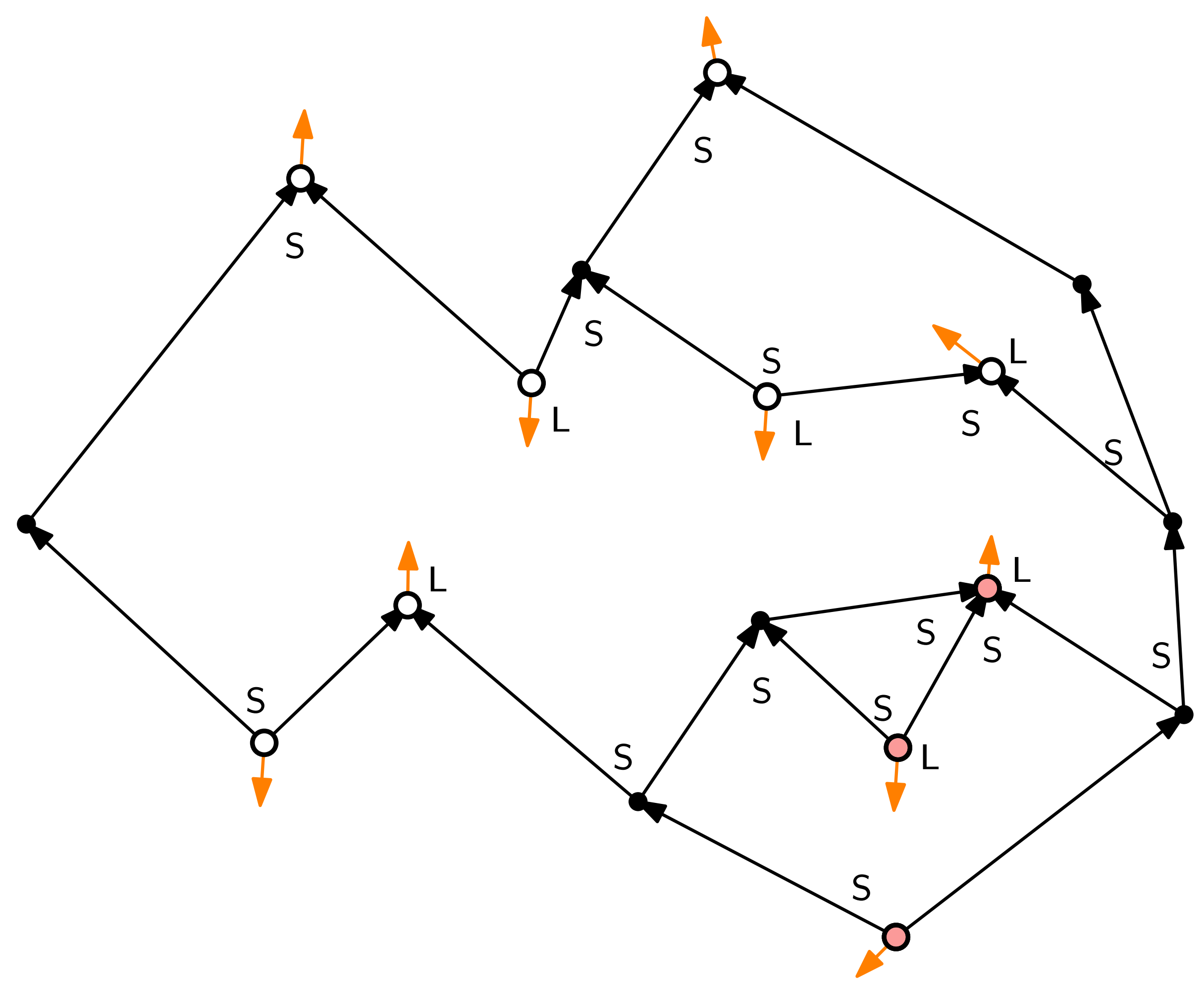

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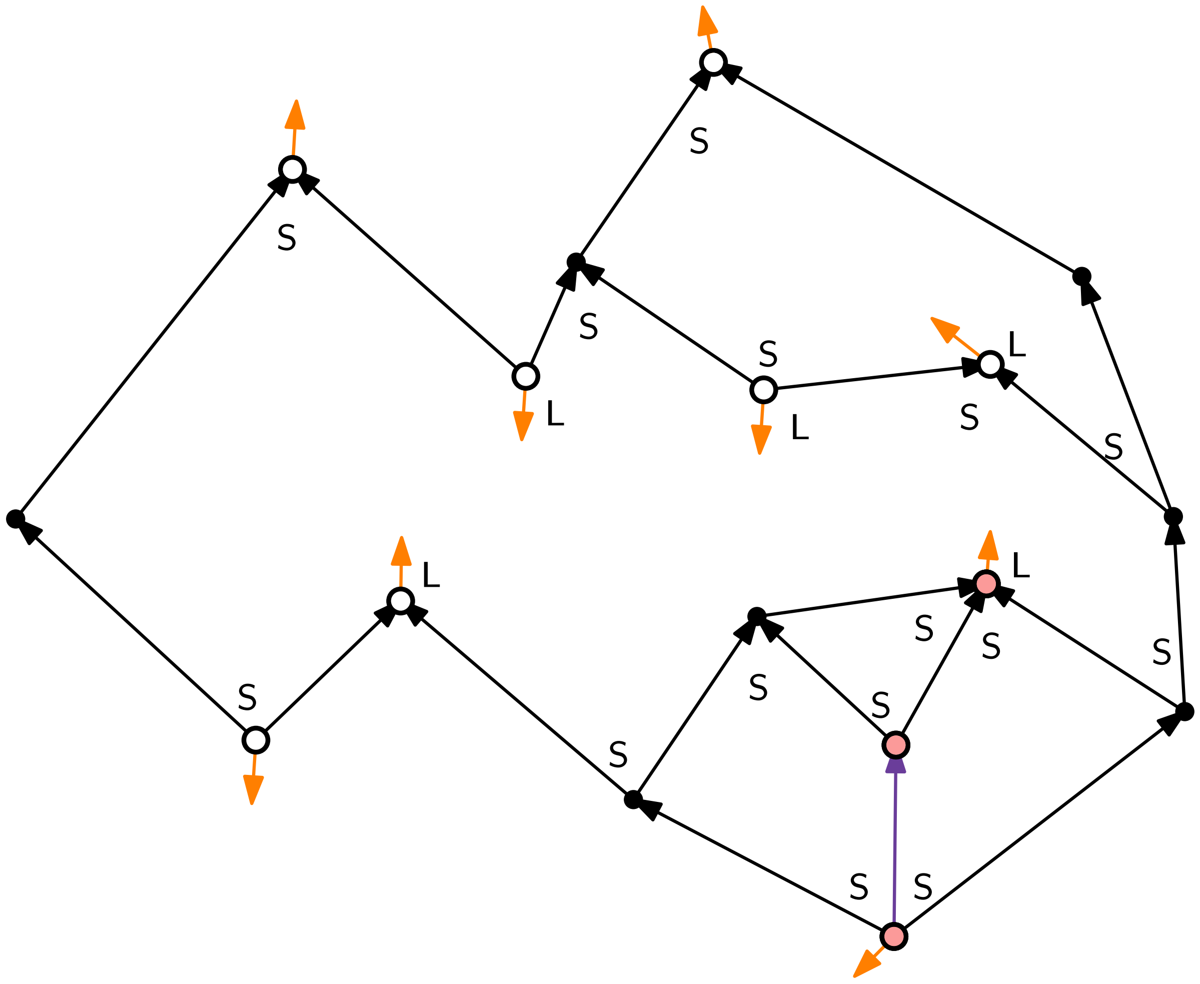

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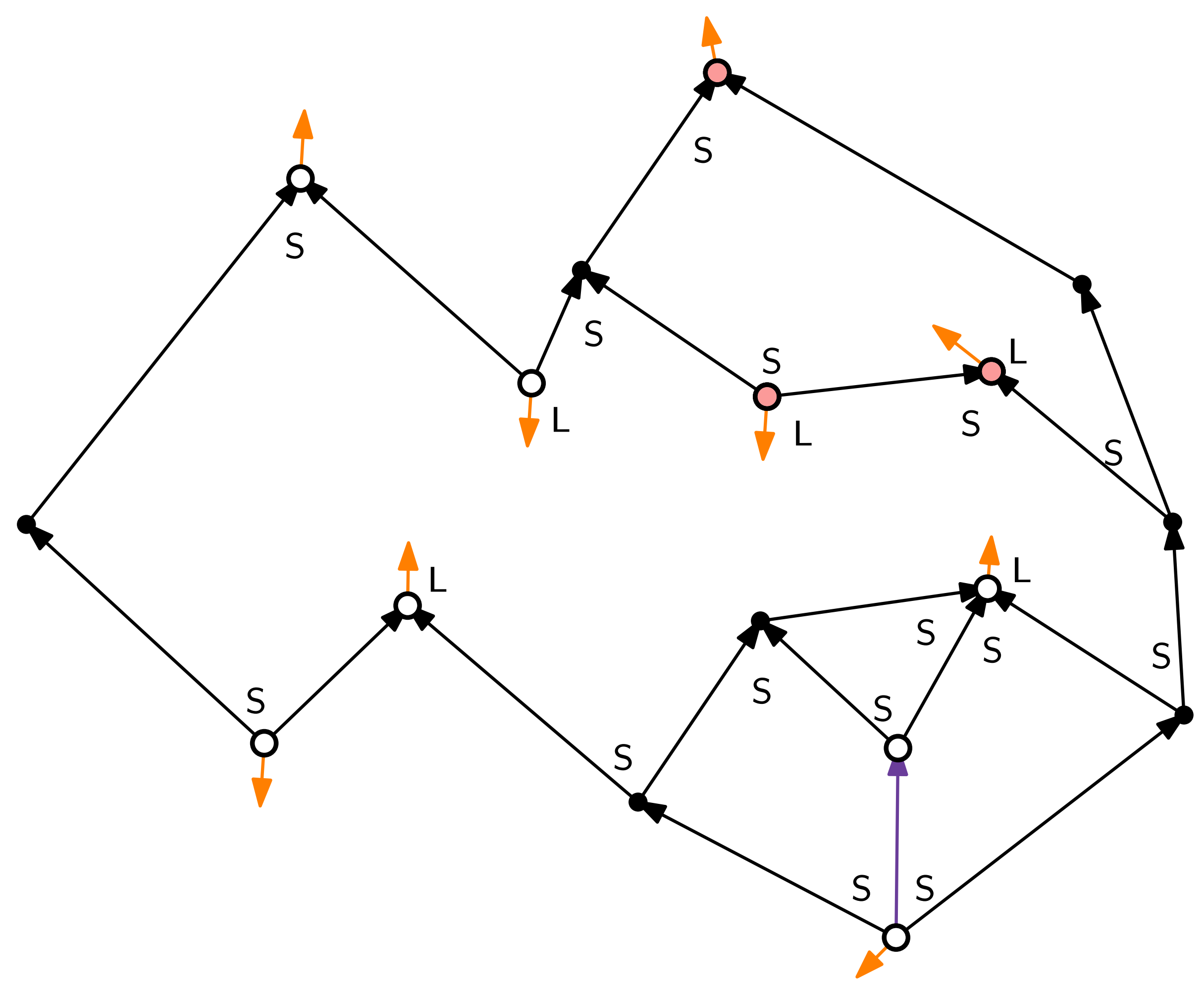

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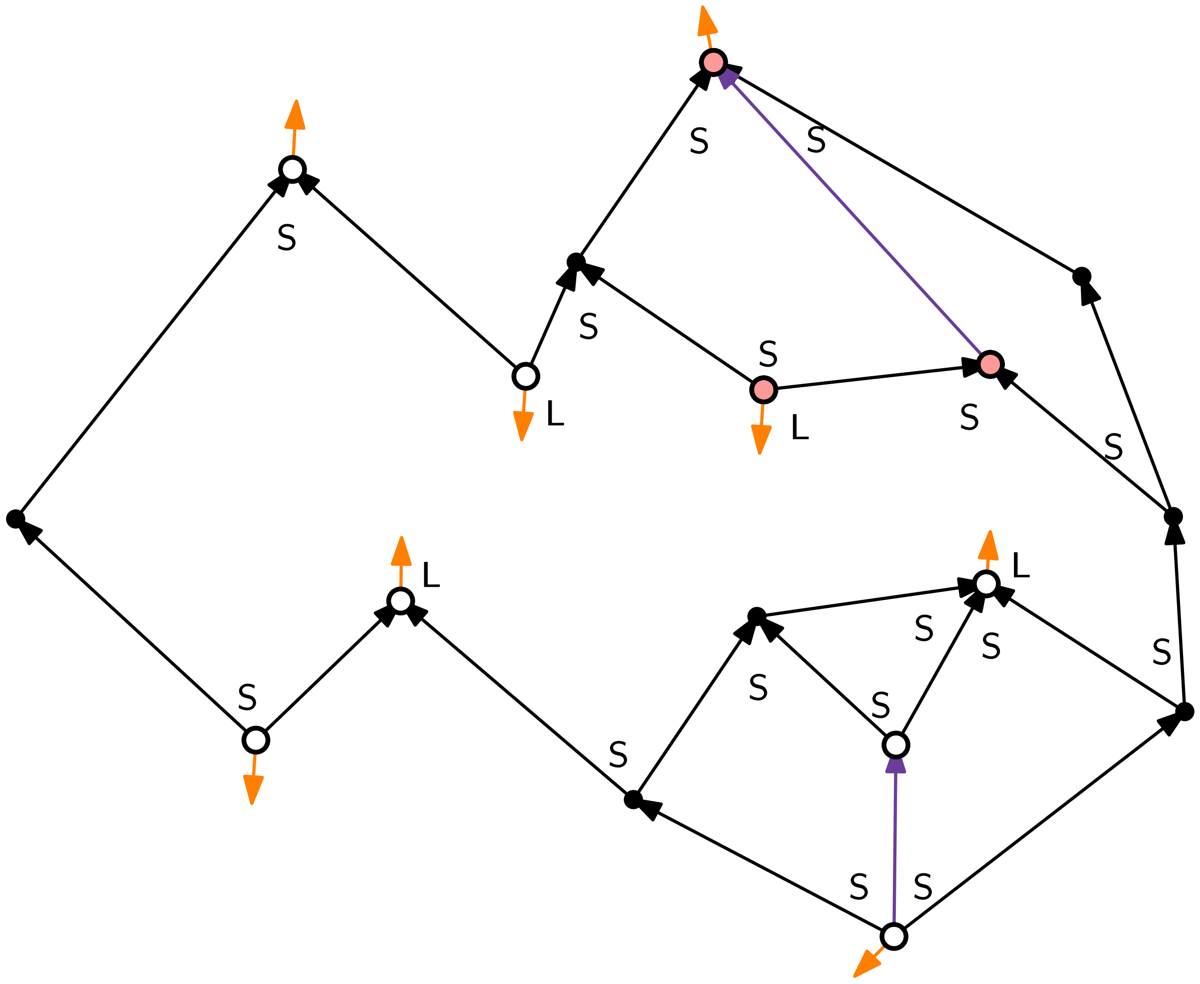

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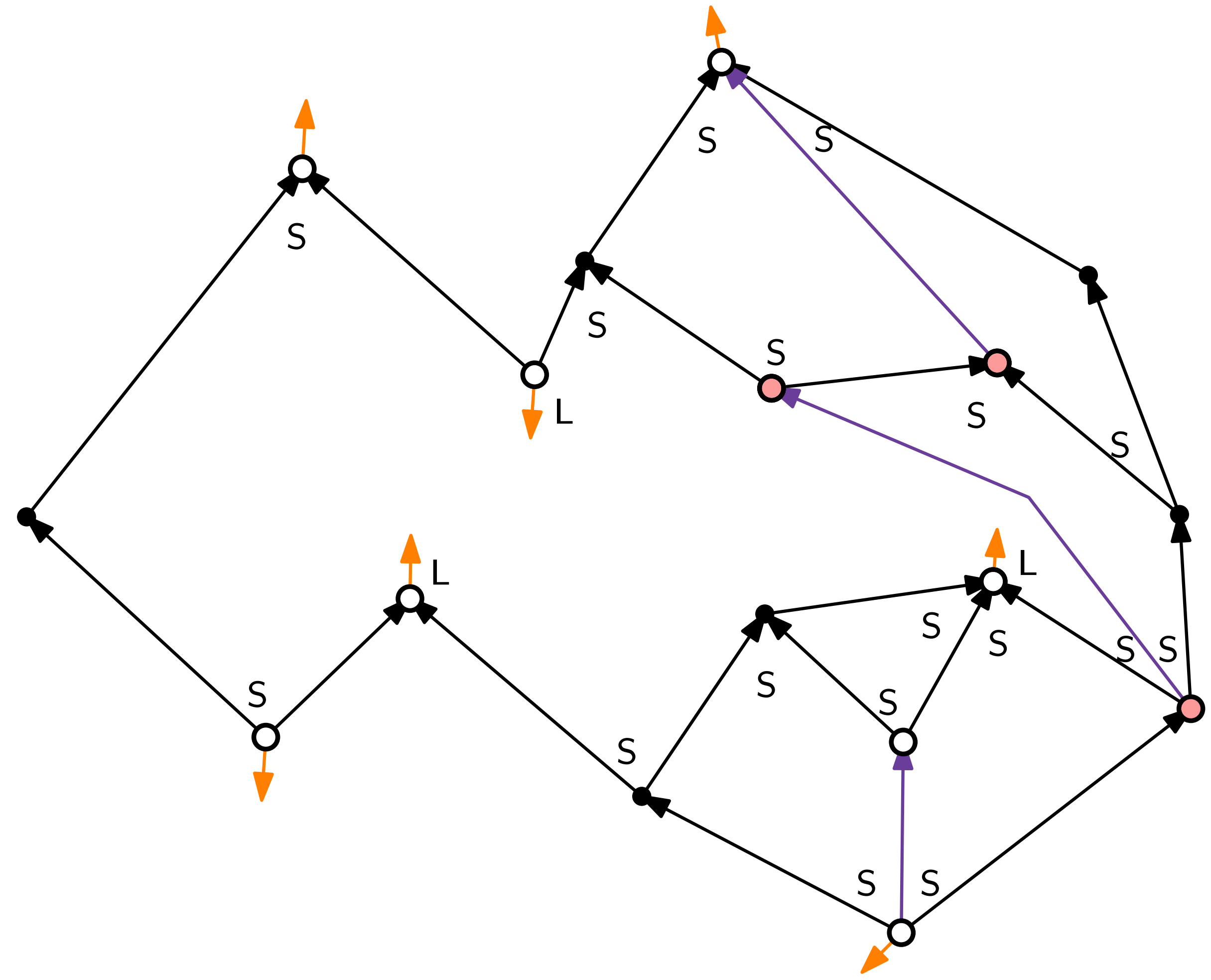

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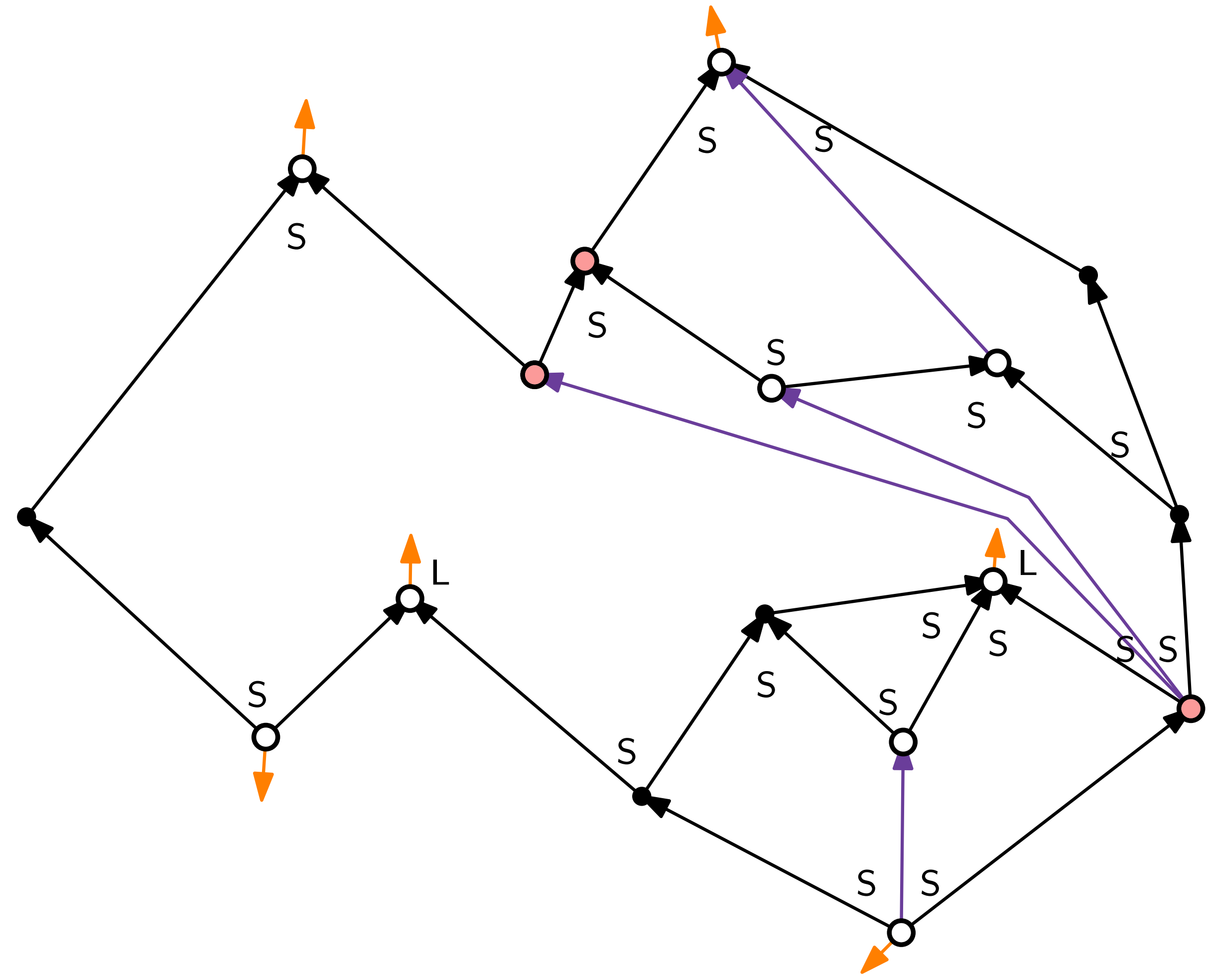

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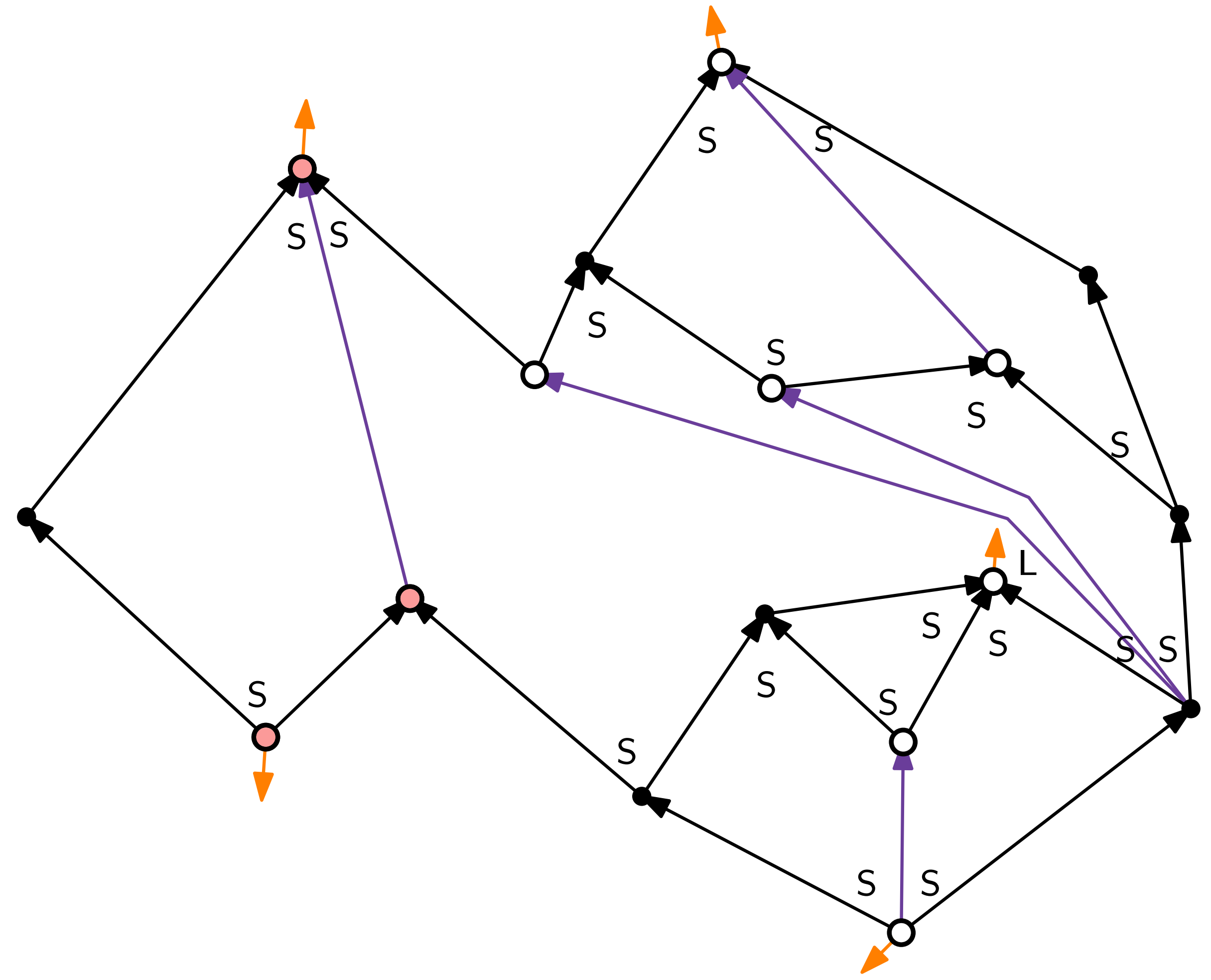

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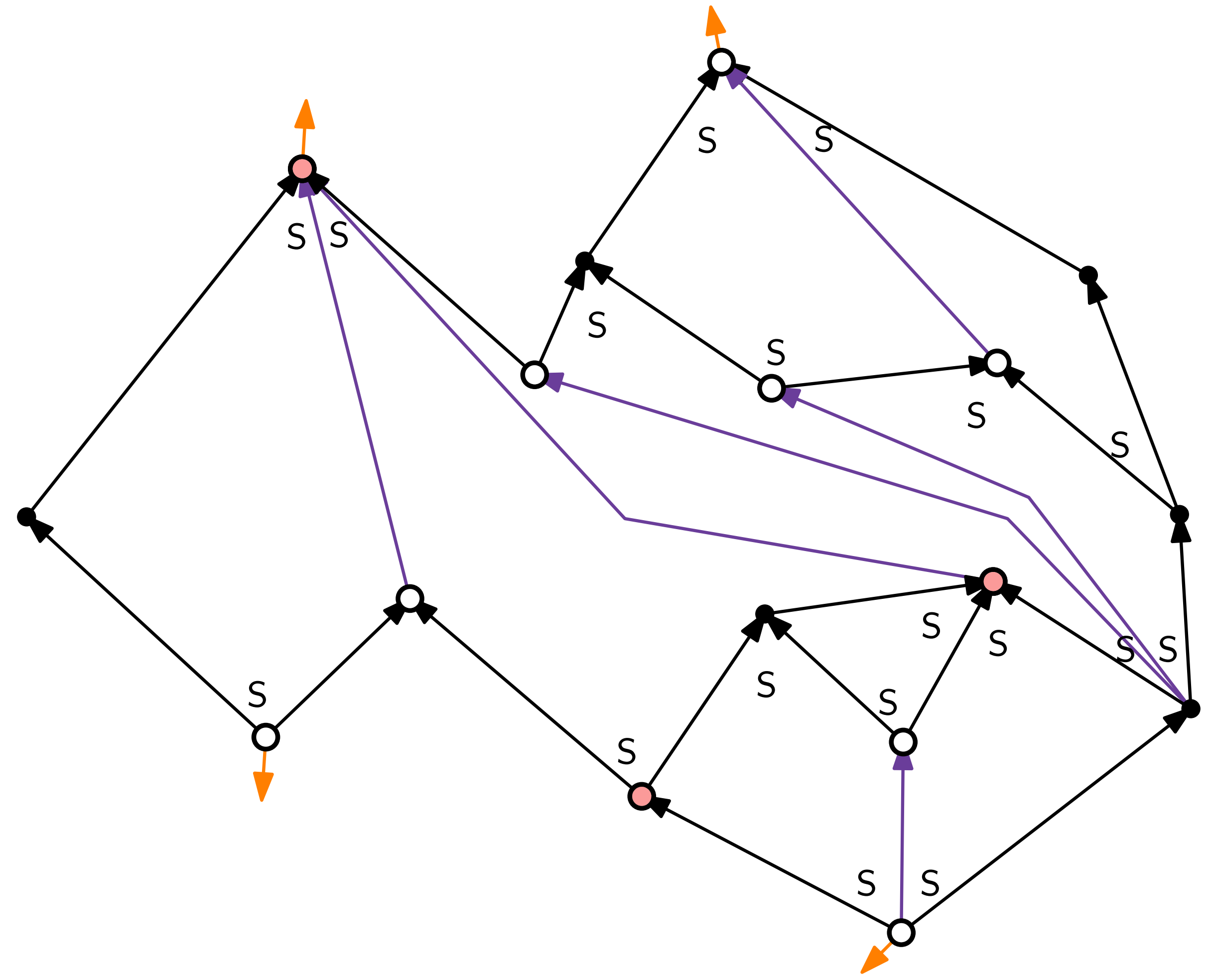

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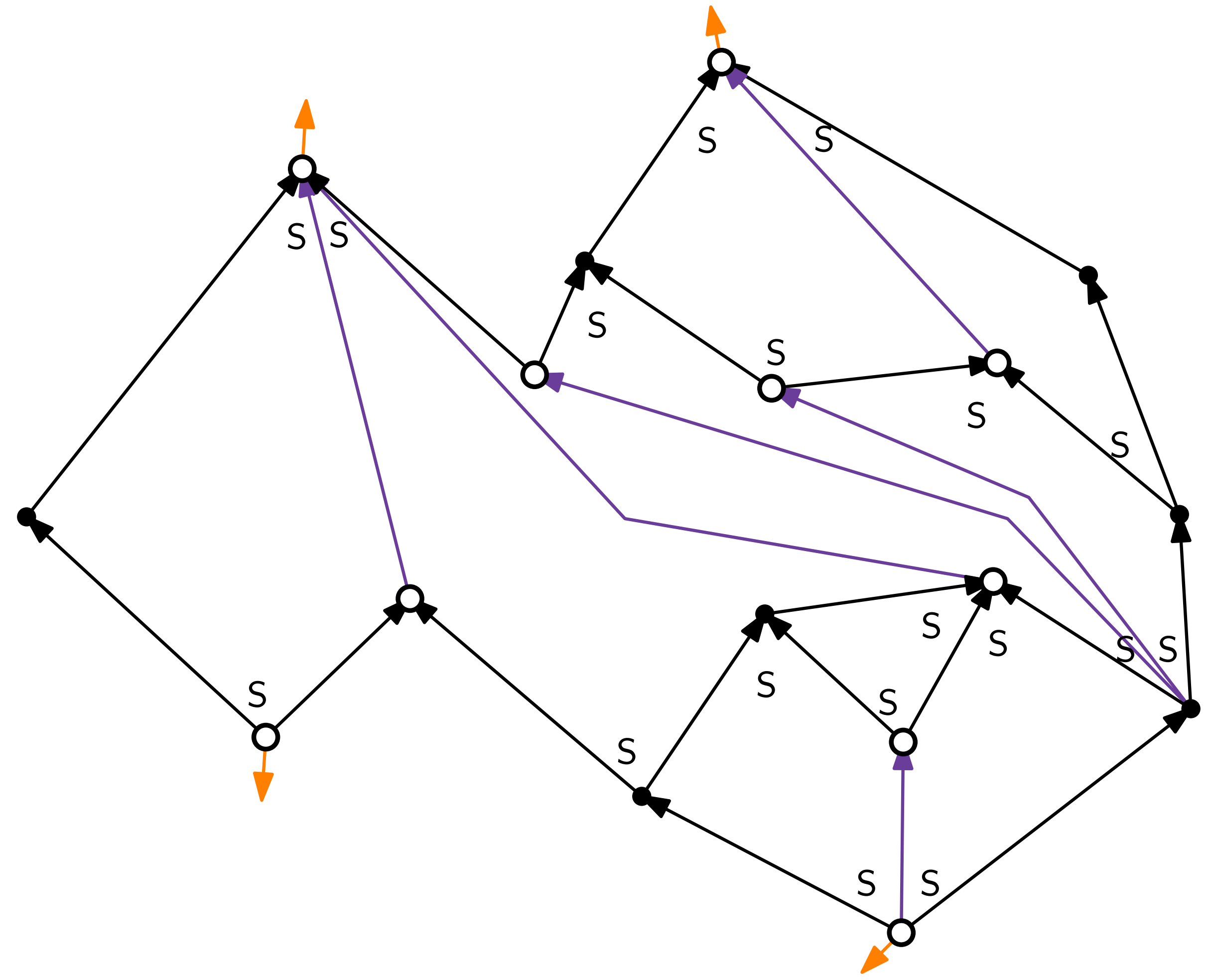

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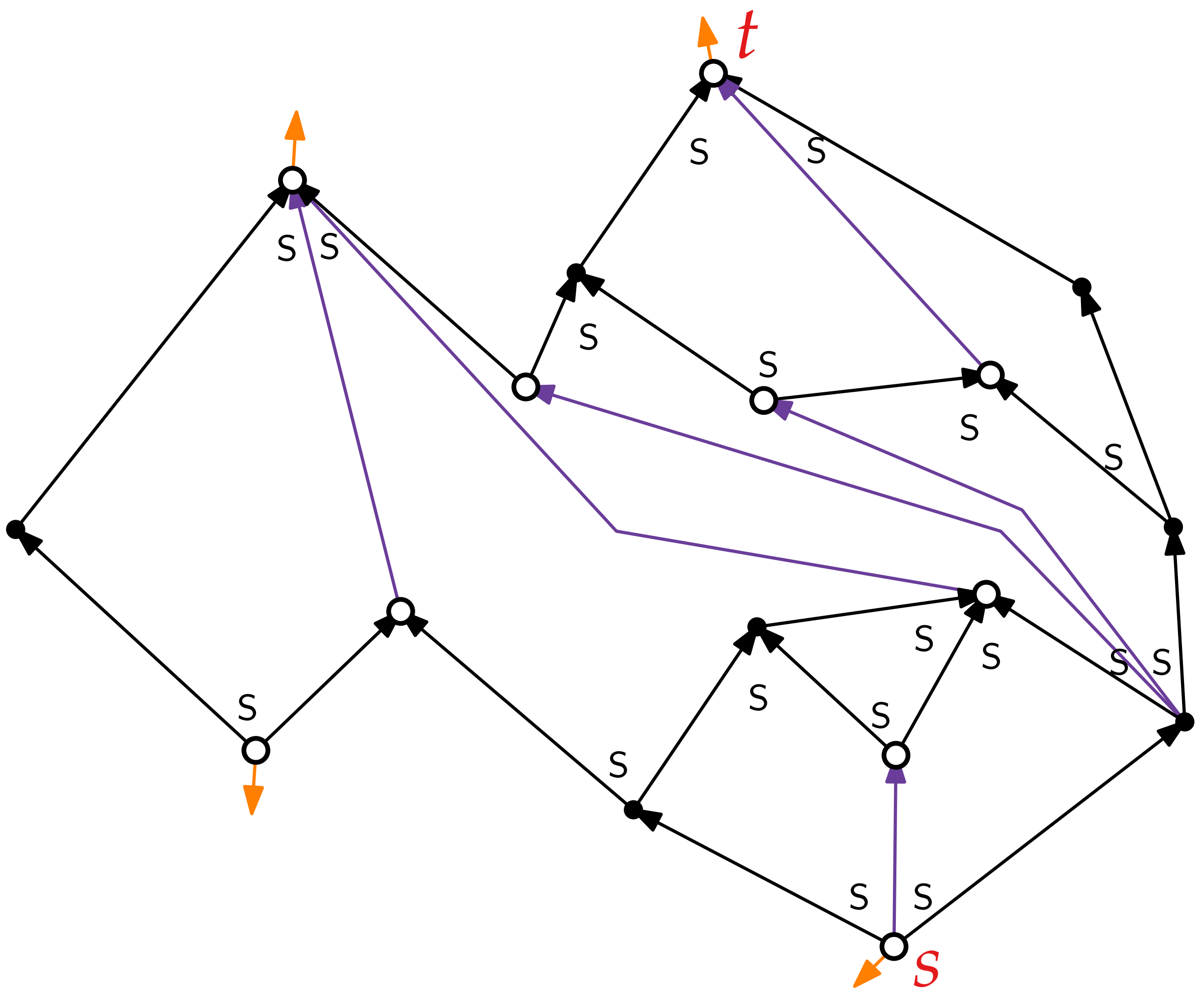

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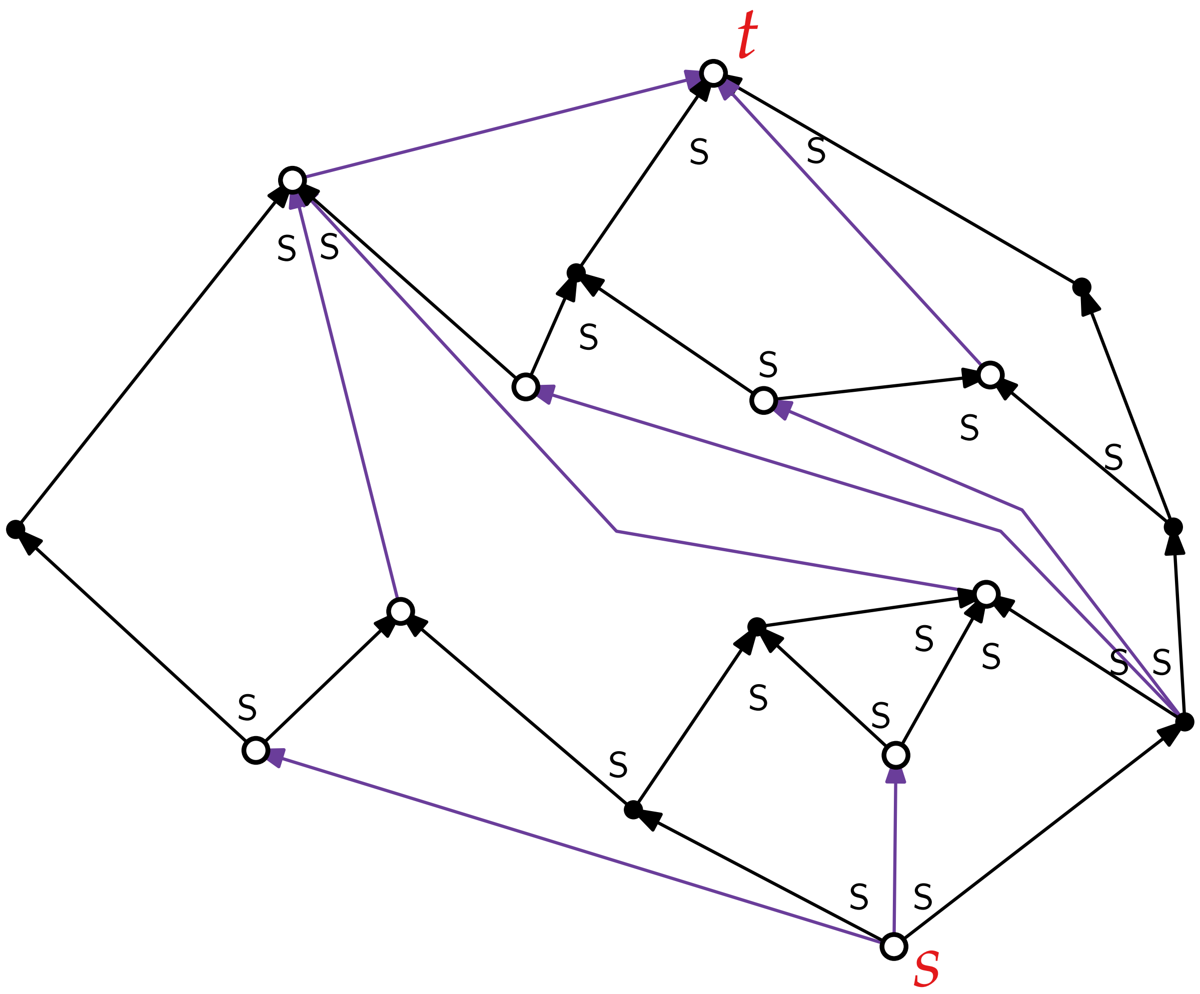

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- If $G$ bimodal and $\Phi$ exists, refine $G$ to plane st-digraph $H$.
- Draw $H$ upward planar.
- Deleted edges added in refinement step.

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Flow $(v, f) = 1$ from global source/sink $v$ to the incident face $f$ its large angle gets assigned to.

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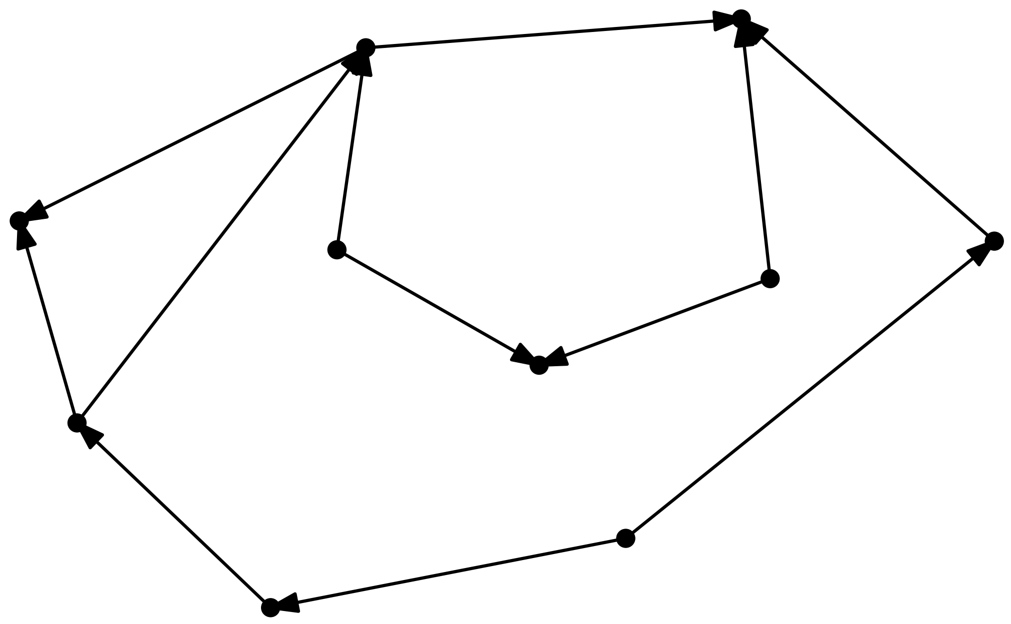

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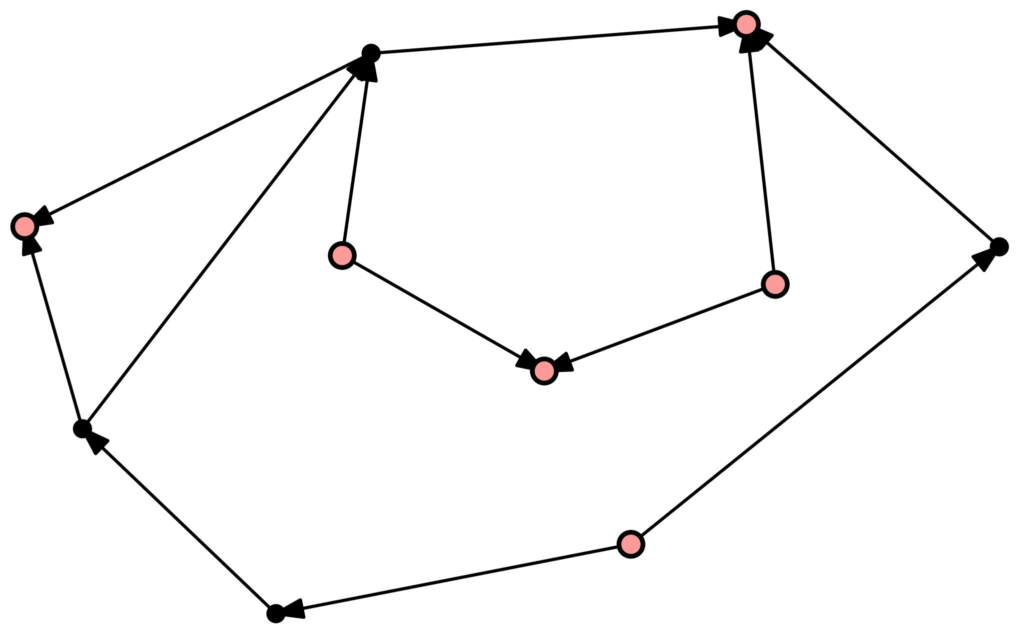

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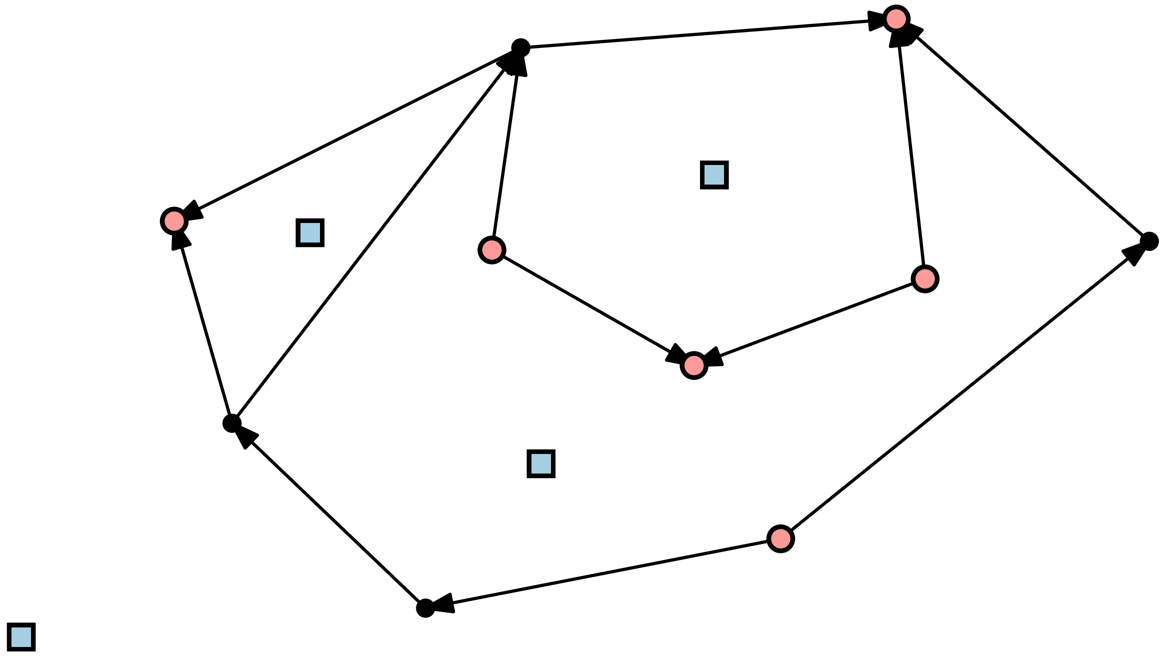

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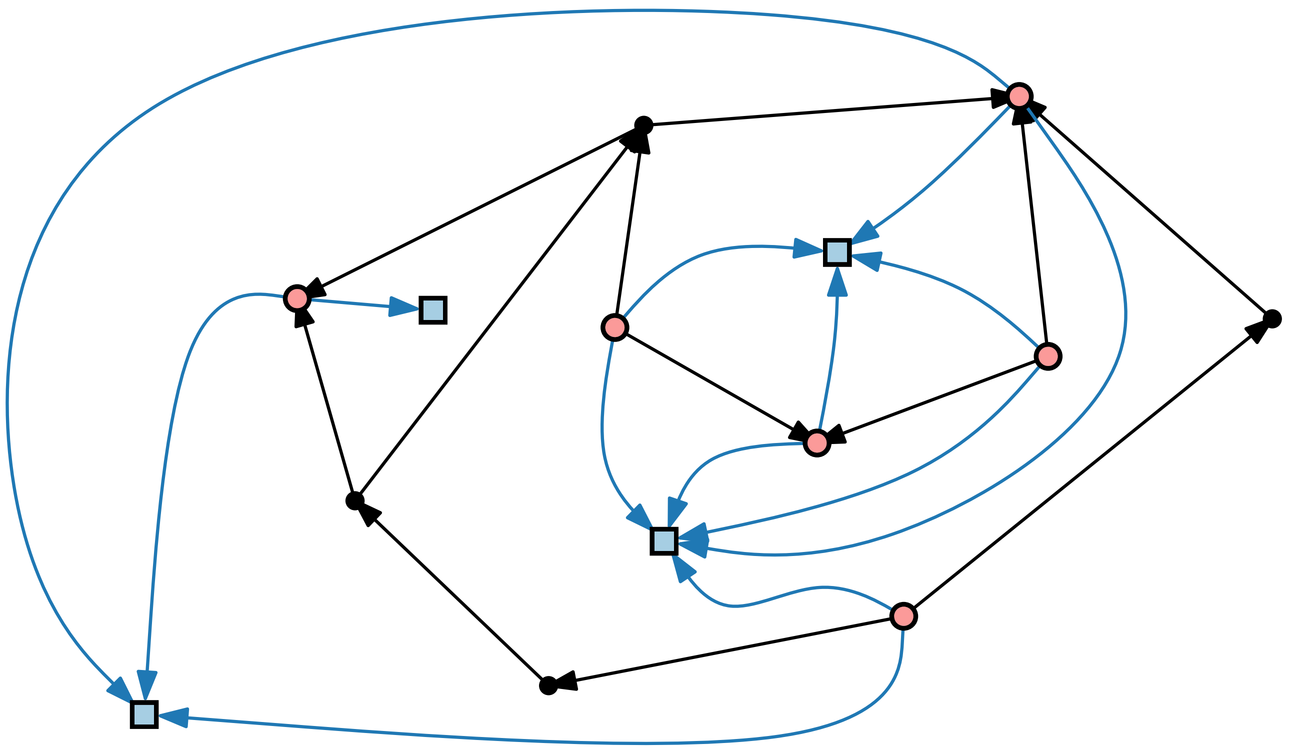

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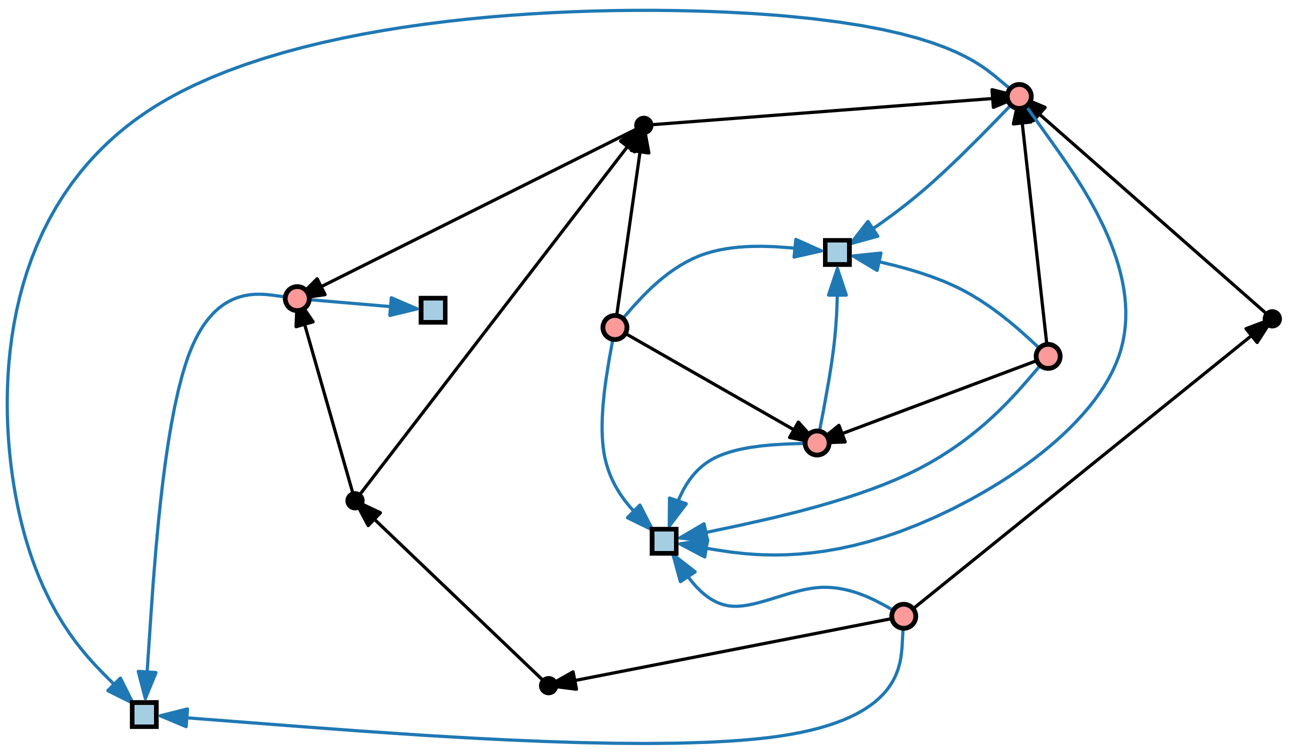

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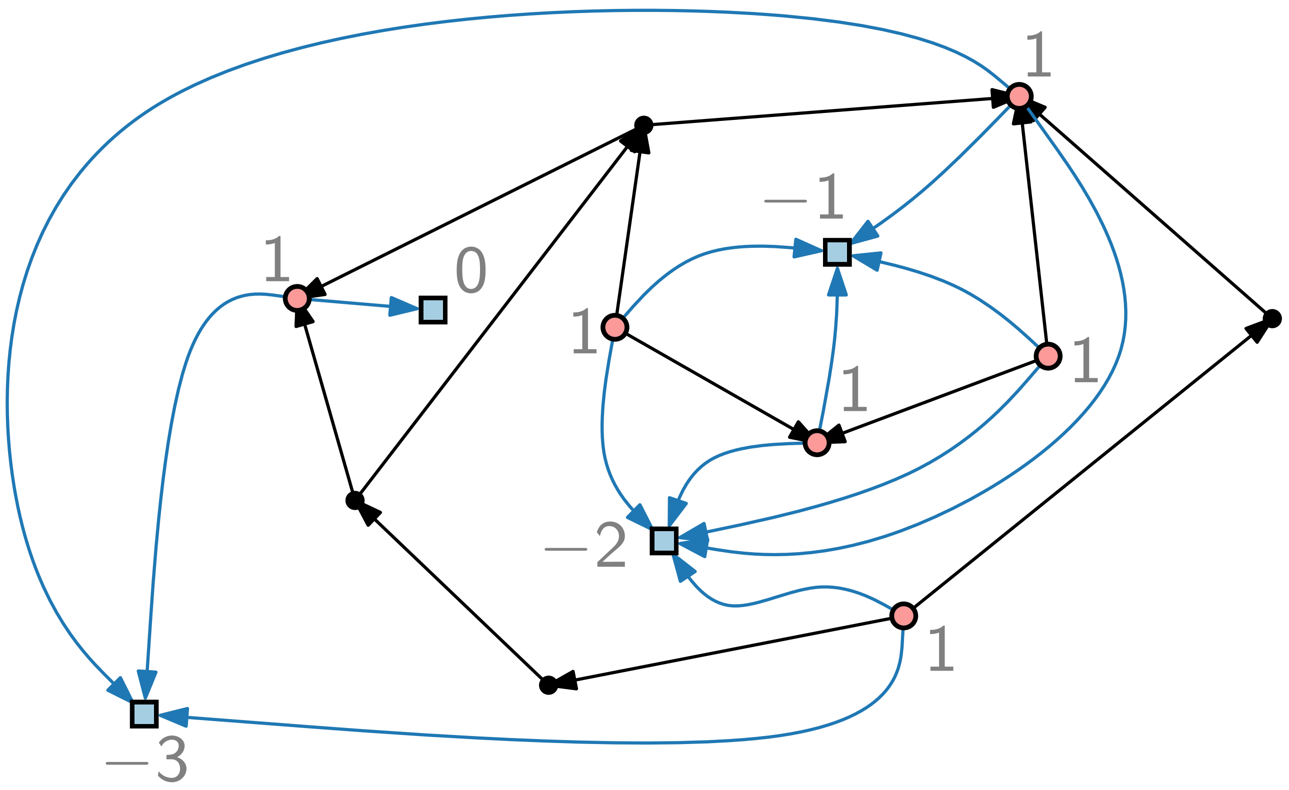
- $E' = \{(v, f) \mid v \text{ incident to } f\}$
- $\ell(e) = 0 \;\forall e \in E'$
- $u(e) = 1 \;\forall e \in E'$
- $d(p) = \begin{cases} 1 & \forall p \in W \cap V \\ -(A(p) - 1) & \forall p \in F \setminus \{f_0\} \\ -(A(p) + 1) & p = f_0 \end{cases}$

**Example.**



# Finding a consistent assignment

**Idea.**
Flow $(v, f) = 1$ from global source/sink $v$ to the incident face $f$ its large angle gets assigned to.

**Flow network.**

$N_{F,f_0}(G) = ((W, E'); \ell; u; d)$

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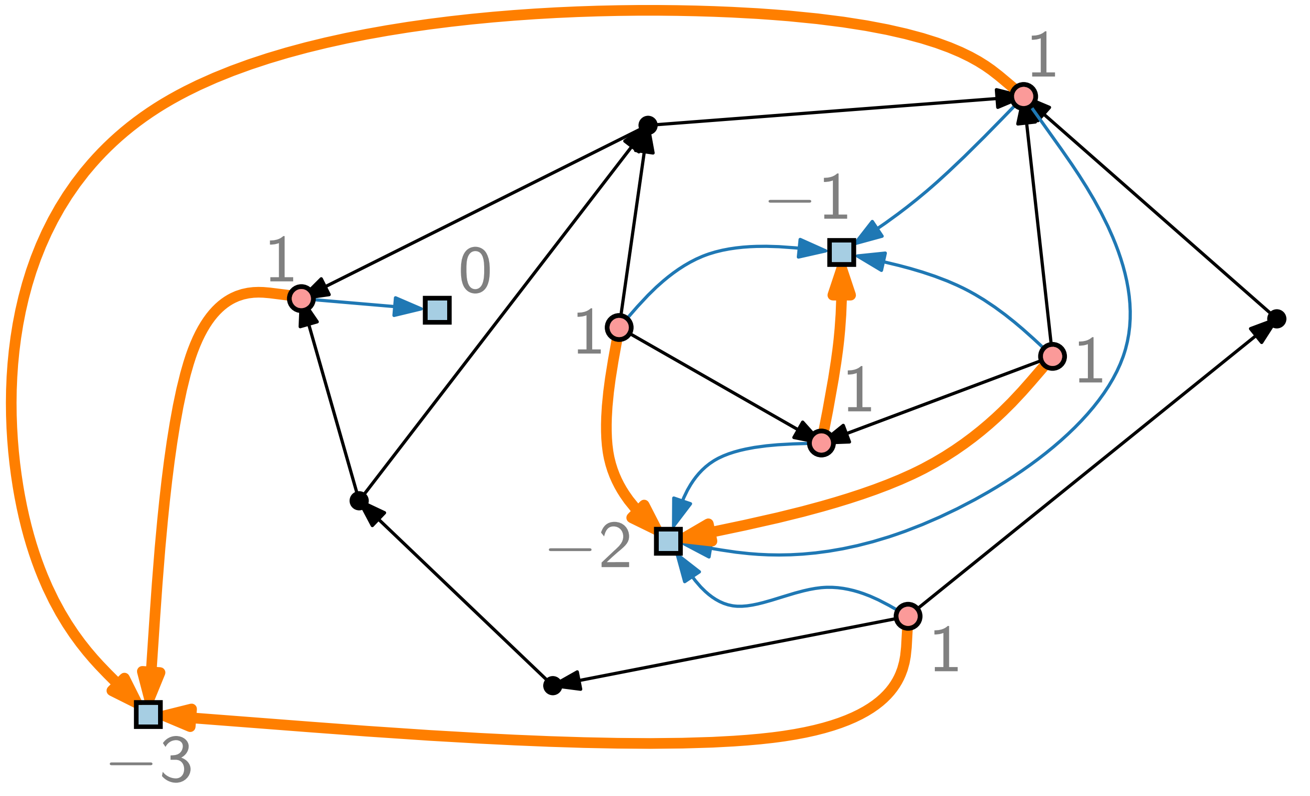

# Discussion

- There exist fixed-parameter tractable algorithms to test upward planarity of general digraphs with the parameter being the number of triconnected components.
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# Discussion

- There exist fixed-parameter tractable algorithms to test upward planarity of general digraphs with the parameter being the number of triconnected components.
  [Healy, Lynch 2005, Didimo et al. 2009]
- Finding assignment in Theorem 2 can be sped up to $\mathcal{O}(n + r^{1.5})$ where $r = \#$ sources/sinks.
  [Abbasi, Healy, Rextin 2010]
- Many related concepts have been studied: quasi-planarity, upward drawings of mixed graphs, upward planarity on cyclinder/torus, ...

# Literature

- [GD Ch. 6] for detailed explanation

Orginal papers referenced:


- [Kelly '87] Fundamentals of Planar Ordered Sets
- [Di Battista, Tamassia '88] Algorithms for Plane Representations of Acyclic Digraphs
- [Garg, Tamassia '95] On the Computational Complexity of Upward and Rectilinear Planarity Testing
- [Hutton, Lubiw '96] Upward Planar Drawing of Single-Source Acyclic Digraphs
- [Bertolazzi, Di Battista, Mannino, Tamassia '94] Upward Drawings of Triconnected Digraphs
- [Healy, Lynch '05] Building Blocks of Upward Planar Digraphs
- [Didimo, Giardano, Liotta '09] Upward Spirality and Upward Planarity Testing
- [Abbasi, Healy, Rextin '10] Improving the running time of embedded upward planarity testing